Periodic and Solitary Traveling Wave Solutions for the Generalized Kadomtsev-Petviashvili Equation, II

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Abstract. As a continuation of our previous work we improve here some results on convergence of periodic KP traveling waves to solitary ones as period goes to infinity. In addition we present some qualitative properties of such waves as well as nonexistence results in the case of general nonlinearities. We suggest here an approach which does not use any scaling argument.

Key words: generalized Kadomtsev-Petviashvili equation, traveling waves, variational methods

AMS subject classification: 35Q53, 35B10, 35A35, 35A15
1 Introduction

Kadomtsev-Petviashvili equations, both original and generalized, appear in the theory of weakly nonlinear dispersive waves [7]. They read

\[
\begin{align*}
    u_t + u_{\xi \xi \xi} + f(u)_{\xi} + \varepsilon v_y &= 0 \\
    v_\xi &= u_y
\end{align*}
\]

or, eliminating \(v\),

\[
    (u_t + u_{\xi \xi \xi} + f(u)_{\xi})_\xi + \varepsilon u_{yy} = 0 .
\]

More precisely, these are KP-I equations if \(\varepsilon = -1\) and KP-II equations if \(\varepsilon = +1\). The original KP equations correspond to the case \(f(u) = \frac{1}{2}u^2\) and form a completely integrable Hamiltonian system and were studied extensively by means of algebro-geometrical methods (see e.g. [8]). There is also a number of papers dealing with more general equations (1) or (2) mainly in the case of power nonlinearity: [1, 2, 3, 4, 5, 10, 17, 21, 24] to mention a few. In particular, solitary traveling waves were studied [1, 2, 3, 4, 5, 10, 17, 24]. Here we consider mainly the case of KP-I equations. Remark that KP-II equations do not possess solitary traveling waves at all (see [5] for the case of power nonlinearity and Section 4 below).

The present paper is a direct continuation of our previous work [17]. It concerns the existence of ground traveling waves both periodic and solitary and the limit behavior of periodic waves as period goes to infinity. Corresponding equations for traveling waves read

\[
\begin{align*}
    -cu_x + u_{xxx} + f(u)_x + \varepsilon v_y &= 0 \\
    v_x &= u_y
\end{align*}
\]

and

\[
    (-cu_x + u_{xxx} + f(u)_x)_x + \varepsilon u_{yy} = 0 ,
\]

respectively. Here \(x = \xi - ct\) (for \(c > 0\) is the wave speed. In [17] we have proved that \(k\)-periodic in \(x\) ground waves converge to a solitary ground wave in a very strong sense (Theorem 5 of that paper). Unfortunately, that result does not cover the case of original KP equation while includes the case \(f(u) = u^3\). And the first aim of the present paper is to extend the results of Theorems 4 and 5 [17] in order to include nonlinearity.
like \( f(u) = |u|^{p-1} \Gamma 2 < p < 6 \). This will be done in Section 2. Our second goal is to discuss some qualitative properties of KP traveling waves. Here we follow very closely the paper [6] and point out only the main differences. Finally we discuss nonexistence of traveling waves both solitary and periodic for general nonlinearities.

All the assumptions we impose here are satisfied for the nonlinearities
\[
f(u) = |u|^{p-1} \text{ and } f(u) = c|u|^{p-2}u + \sum_{i=1}^{k} c_i |u|^{p-2}u \Gamma \text{ with } c, c_i > 0 \Gamma 2 < p < 6 \Gamma 2 < p_i < p.
\]
Unfortunately the Assumptions (N) and (N1) below are not satisfied for a very interesting nonlinearity \( f(u) = u^2 - u^3 \).

## 2 Ground Waves

Denote by
\[
F(u) = \int_0^u f(t)dt
\]
the primitive function of \( f \). We make the following assumptions:

1° \( f \in C(\mathbb{R}) \), \( f(0) = 0 \);

2° \( |f(u)| \leq C(1 + |u|^{p-1}) \), \( 2 < p < 6 \), and \( f(u) = o(|u|) \) as \( u \to 0 \);

3° there exists \( \varphi \in C_0^\infty(\mathbb{R}^2) \) such that
\[
\frac{1}{\lambda^2} \int_{\mathbb{R}^2} F(\lambda \varphi_x) \to +\infty
\]
as \( \lambda \to +\infty \);

4° there exists \( \mu > 2 \) such that \( \mu F(u) \leq uf(u) \) for all \( u \in \mathbb{R} \).

Remark that if \( F(u) > 0 \) for all \( u \neq 0 \) then 3° follows from 4°.

Let \( Q_k = (-k/2, k/2) \times \mathbb{R} \Gamma 0 < k \leq \infty \). We set
\[
D_{x,k}^{-1} u(x,y) = \int_{-k/2}^x u(s,y)ds, \quad k \in (0, \infty].
\]
(5)

We shall write simply \( D_x^{-1} \) in the case \( k = \infty \). Define the Hilbert space \( X_k \) as the completion of \( \{ \varphi_x : \varphi \in C_k^\infty \} \) where \( C_k^\infty \) is the space of smooth functions on \( \mathbb{R}^2 \) which are \( k \)-periodic in \( x \) and have finite support in \( y \) with respect to the norm \( \|u\|_k = (u, u)_{k}^{1/2} \Gamma \)

\[
(u, v)_k = \int_{Q_k} u_x v_x + D_{x,k}^{-1} u_y \cdot D_{x,k}^{-1} v_y + cu v.
\]
Similarly $\Gamma \{ X = X_\infty \}$ is the completion of $\{ \varphi_x : \varphi \in C^\infty_0(\mathbb{R}^2) \}$ with respect to the norm $\| u \| = \| u \|_\infty = (u,u)_1^{1/2} = (u,u)_2^{1/2}$. The operator $D^{-1}_{x,k}$ is well-defined on the space $X_\infty \Gamma k \in (0, \infty)$.

For $k$-periodic traveling waves $k \in (0, \infty)$ equation (4) may be written in the form [17]

$$(-u_{xx} + D^{-2}_{x,k} u_{yy} + cu - f(u))_x = 0. \quad (6)$$

Solitary waves are solutions of the same equation (6) with $k = \infty$. The action functional associated with (6) reads [17]

$$J_k(u) = \frac{1}{2} \| u \|_k^2 - \int_{Q_k} F(u); \quad (7)$$

$J_k$ is of the class $C^1$ on $X_k$. We consider weak solutions of (6) i.e. critical points of $J_k$ in $X_k$.

Now let us consider the so-called Nehari functional

$$I_k(u) = \langle J'_k(u), u \rangle = \| u \|_k^2 - \int_{Q_k} u f(u), \quad (8)$$

and the Nehari manifold

$$S_k = \{ u \in X_k : J_k(u) = 0, u \neq 0 \}.$$

All traveling wave solutions lie in the corresponding Nehari manifold and we will find ground waves i.e. solutions with minimal action among all nontrivial solutions solving the following minimization problem

$$m_k = \inf \{ J_k(u) : u \in S_k \}. \quad (9)$$

Remark that

$$J_k(u) = \int_{Q_k} \frac{1}{2} u f(u) - F(u), \quad u \in S_k. \quad (10)$$

In what follows we will omit the subscript $k$ if $k = \infty$ and write simply $J, I, \ldots$.

Throughout this section in addition to Assumptions 1–4 we impose the following one

3
(N) For any \( u \in L^2(\mathbb{R}^2) \) such that
\[
\int_{\mathbb{R}^2} uf(u) > 0 ,
\]
the function of \( t \)
\[
t^{-1} \int_{\mathbb{R}^2} uf(tu)
\]
is strictly increasing on \((0, +\infty)\).

In the proof of Theorem 11\cite{[17]} we have introduced the Mountain Pass Values \( c_k \) for \( J_k \) and proved that they are uniformly bounded from below and above by positive constants. More precisely\cite{[17]}
\[
c_k = \inf_{\gamma \in \Gamma_k} \max_{t \in [0,1]} J_k(\gamma(t)) ,
\]
where
\[
\Gamma_k = \{ \gamma \in C([0,1], X_k) : \gamma(0) = 0, J_k(\gamma(1)) < 0 \} .
\]
Here we have defined \( \Gamma_k \) in a slightly different way than in [17] but it does not effect on the value of \( c_k \). Consider also another minimax value
\[
\hat{c}_k = \inf_{v \in X_k^+} \sup_{t > 0} J_k(tv) ,
\]
where
\[
X_k^+ = \{ v \in X_k : \int_{Q_k} F(v) > 0 \} .
\]
Due to Assumption 4\cite{[17]} \( X_k^+ \neq \emptyset \).

**Lemma 1** For every \( v \in X_k^+ \) there exists a unique \( t_k = t_k(v) \) such that \( t_k v \in S_k \),
\[
J_k(t_k v) = \max_{t > 0} J_k(tv) ,
\]
and \( t_k(v) \) depends continuously on \( v \in X_k^+ \).

**Proof.** Assumption 4 implies that
\[
\int_{Q_k} vf(v) > 0
\]
for any $v \in X_k^+$. Therefore, due to Assumption (N), the function

$$\frac{d}{dt} J_k(tv) = I_k(tv) = t^2(\|v\|_k^2 - t^{-1} \int_{Q_k} v f(tv))$$

vanishes at only one point $t_k = t_k(v) > 0$. Equation (10) and Assumption 4 imply that $J_k$ is positive on $S_k$. Since $J_k(0) = 0$, we see that $t_k$ is a point of maximum for $J_k(tv)$. Continuity of $t_k(v)$ is easy to verify. □

Lemma 2 $c_k = c_k' = m_k$.

Proof. Since $uf(u)$ is subquadratic at 0 and the quadratic part of $J_k$ is positive defined, we see that $I_k(v) > 0$ in a neighborhood of the origin, except of 0. Hence, $I_k(\gamma(t)) > 0$ for all $\gamma \in \Gamma_k$ for small $t > 0$. Due to Assumption 4, for $v \in X_k^+$, we have

$$2J_k(v) = \|v\|_k^2 - 2 \int_{Q_k} F(v) > \|v\|_k^2 - \mu \int_{Q_k} F(v) \geq \|v\|_k^2 - 2 \int_{Q_k} v f(v) = I_k(v).$$

Hence, $I_k(\gamma(1)) < 0$. Therefore, $\gamma(t)$ crosses $S_k$ and this implies that $c_k \geq m_k$.

By Assumption 4, for any $v \in X_k^+$, we have $F(tv) \geq \alpha t^\gamma > 0$ if $t > 0$ is large enough. This implies that $J_k(tv) < 0$ for every $v \in X_k^+$ and sufficiently large $t > 0$. Hence, the half-axis $\{tv : t > 0\}$ generates in a natural way an element of $\Gamma_k$. This implies the inequality $c_k \leq c_k'$.

Now let $v \in S_k$. By the definition of $I_k$,

$$\sigma = \int_{Q_k} v f(v) > 0,$$

and (N) implies that

$$\frac{d}{dt} \int_{Q_k} F(tv) = f(t^{-1} \int_{Q_k} v f(tv)) \geq t^{-1} \int_{Q_k} v f(tv) \geq \sigma > 0$$
provided $t \geq 1$. Hence for $t > 0$ large enough

$$\int_{Q_t} F(tu) > 0.$$  

By definitions of $c_k$ and $m_k$ we see that $c_k = m_k$. □

**Theorem 1** Assume $1^04^0$ and (N) to be fulfilled. Then, for any $k \in (0, \infty)$, there exists a minimizer $u_k \in S_k$ of (9) which is a critical point of $J_k$. Moreover, $J_k(u_k) = m_k$ is bounded from above and below by positive constants independent on $k$.

**Proof.** In the proof of Theorem 1 it is shown that there exists a Palais-Smale sequence $u_{k,n} \in X_k$ at the level $c_k$. I.e.

$$J_k'(u_{k,n}) \to 0, \quad J_k(u_{k,n}) \to c_k$$

as $n \to \infty$. Moreover $u_{k,n} \rightharpoonup u_k$ weakly in $X_k$ and strongly in $L^p_{loc}(\mathbb{R}^2)$ where $u_k \in X_k$ is a nontrivial solution of (6). Therefore

$$I_k(u_{k,n}) = \langle J_k'(u_{k,n}), u_{k,n} \rangle \to 0$$

and

$$J_k(u_{k,n}) - \frac{1}{2} I_k(u_{k,n}) = \int_{Q_1} \left( \frac{1}{2} u_{k,n} f(u_{k,n}) - F(u_{k,n}) \right) \to c_k.$$

Due to $4^0$ the integrand here is nonnegative and since $u_{k,n} \to u_k$ in $L^2_{loc}(\mathbb{R}^2)$ we have

$$\int_{Q_1} \left( \frac{1}{2} u_k f(u_k) - F(u_k) \right) \leq c_k.$$  

However $u_k$ is a nontrivial solution hence $u_k \in S_k$. Therefore we deduce from (10) that

$$J_k(u_k) = \int_{Q_1} \left( \frac{1}{2} u_k f(u_k) - F(u_k) \right) \geq m_k.$$  

Now Lemma 2 implies that $J_k(u_k) = m_k$ and $u_k$ is a ground wave solution.

The last statement of the theorem follows immediately from Lemma 2 and uniform estimates for $c_k$. □
Remark 1  The Nehari variational principle suggested in [13] was used successfully in many papers (see e.g. [2, 9, 14, 15, 16, 23]). In all these papers except of [16] the geometry of Nehari manifold is simple enough: it is a bounded surface without boundary around the origin like sphere. In the case we consider here the picture is different: $S_k$ may look like sphere if e.g. $|u|^p - 2|u|_2$ and may be unbounded if e.g. $|u|^p - 1$. Nevertheless in any case $S_k$ separates the origin and the domain of negative values of $J_k$ which is sufficient for our purpose. In [16] such a manifold is also unbounded in general but there we have used different arguments.

Now we are going to study the behavior of $u_k$ as $k \to \infty$. Recall the definition of cut-off operators $P_k : X_k \to X [17]$. Let $\chi_k \in C_0^\infty(\mathbb{R})$ be a nonnegative function such that $\chi_k(x) = 1$ for $x \in [−k/2, k/2] \Gamma \chi_k(x) = 0$ for $|x| \geq (k + 1)/2 \Gamma$ and $|\chi_k''| \leq C_0 \Gamma$ with some constant $C_0 > 0$. We set

$$P_ku(x, y) = [\chi_k(x)D_{x,k}^{-1}u(x, y)]_x.$$  

Theorem 2  Assume that $1^0 - 4^0$ and (N) are satisfied. Let $u_k \in X_k$ be a sequence of ground wave solutions. Then there exist a nontrivial ground wave $u \in X$ and a sequence of vectors $\zeta_k \in \mathbb{R}^2$ such that, along a subsequence, $P_ku_k(\cdot + \zeta_k) \to u$ weakly in $X$. If in addition

$$|f(u + v) - f(u)| \leq C(1 + |u|^{p-2} + |v|^{p-2})|v|, \quad v \in \mathbb{R}, \quad (11)$$

then, along the same subsequence,

$$\lim_{k \to \infty} \|u_k(\cdot + \zeta_k) - u\|_k = 0.$$

Proof.  By Theorem 2 [17] there is a nontrivial solution $u \in X$ such that $P_ku_k(\cdot + \zeta_k) \to u$ weakly in $X$ for some $\zeta_k \in \mathbb{R}^2$ (along a subsequence). Let us prove that $u$ is a ground wave i.e.

$$J(u) = \inf\{J(v) : v \in S\} = m.$$

First of all for any $v \in S$ and any $\varepsilon > 0$ there exist $k_\varepsilon$ and $v_k \in S_k$ such that

$$J_k(v_k) \leq J(v) + \varepsilon, \quad k \geq k_\varepsilon.$$
Indeed since $J$ and $I$ are continuous we can find $\varphi_k \in C^\infty_0(Q_k)$ such that $\eta_k = D_x \varphi_k \to v$ in $X$ and hence

$$J(\eta_k) \to J(v), \quad I(\eta_k) \to I(v) = o.$$ 

Since $I(v) = 0$ and $v \neq 0$ we have

$$\int_{Q_k} v f(v) = \|v\|^2 > 0.$$ 

Hence

$$\int_{Q_k} \eta_k f(\eta_k) > 0$$

for $k$ large enough. Due to (N) there exists $\tau_k > 0$ such that $I(\tau_k \eta_k) = 0$ and $\tau_k \to 1$. Let $v_k$ be a unique $k$-periodic function which coincides with $\tau_k \eta_k$ on $Q_k$. Then

$$J_k(v_k) = J(\tau_k \eta_k) \leq J(v) + \varepsilon$$

provided $k$ is large enough.

In particular we have

$$\limsup_{k \to \infty} m_k \leq m.$$ 

Now exactly as in the proof of Theorem 5.17 we see that

$$\liminf_{k \to \infty} m_k \geq J(u) \geq m.$$ 

Hence $m = J(u)$ and $u$ is a ground wave solution.

The second part of the theorem follows from Theorem 3.17 exactly as at the end of proof of Theorem 5.17. □

## 3 Qualitative Properties of Traveling Waves

Now we are going to study such properties of KP traveling waves as symmetry, regularity and decay. We start with the following

**Lemma 3** Under Assumptions 1$^0$ and 2$^0$ any traveling wave is continuous. Moreover, solitary (resp. periodic) wave tends to zero as $(x, y) \to \infty$ (resp. $y \to \infty$).
Proof. For such a wave \( u \in X_k \Gamma \) we have
\[
-\epsilon v_{xx} - v_{yy} + v_{xx} = f(u)_{xx} = g_x x,
\]
Let
\[
(F_{k,x} h)(\xi) = \int_{k/2}^{k/2} h(x) \exp(-i\xi x) dx
\]
be the Fourier transform if \( k = \infty \) (then we write simply \( F_x )\Gamma \) and the sequence of Fourier coefficients if \( k < \infty \). In the last case \( \xi \in (2\pi/k)\mathbb{Z} \). Now we get from (12)
\[
F_{k,x} F_{y} u = p(\xi_1, \xi_2) (F_{k,x} F_{y} g),
\]
where
\[
p(\xi) = p(\xi_1, \xi_2) = \frac{\xi_1^2}{\xi_1^2 + \xi_1^4 + \xi_2^4}.
\]
\( \xi_1 \) and \( \xi_2 \) are dual variables to \( x \) and \( y \Gamma \) respectively. If \( k = \infty \Gamma \) there is nothing to do. The proof of Theorem 1.1\( \Gamma[6] \Gamma \) does not use any particular property of power nonlinearity except of its growth rate.

Now we explain how to cover the case of periodic waves. Recall the following Lizorkin theorem\( \Gamma[11] \). Let \( p(\xi) \Gamma \xi \in \mathbb{R}^n \Gamma \) be of the class \( C^n \) for \( |\xi_j| > 0 \Gamma j = 1, \ldots, n \). Assume that
\[
|\xi_1^{k_1} \cdots \xi_n^{k_n} \frac{\partial^p}{\partial \xi_1^{k_1} \cdots \partial \xi_n^{k_n}}| \leq M,
\]
with \( k_j = 0 \) or \( 1 \Gamma k = k_1 + \cdots + k_n = 0, 1, \ldots, n \). Then \( p(\xi) \) is a Fourier multiplier on \( L^r(\mathbb{R}^n) \Gamma \Gamma < r < \infty \).

We rewrite now (13) as follows
\[
F_{k,x} u = F_y^{-1} [p(\xi_1, \xi_2) F_y F_{k,x} g] = P(\xi_1) g,
\]
where \( P(\xi_1) \) is the operator \( F_y^{-1} p(\xi_1, \cdot) F_y \) for any fixed \( \xi_1 \). It is easy to verify that \( P(\xi_1) \in L(L^r(\mathbb{R}_y)) \Gamma \) the space of bounded linear operators in \( L^r(\mathbb{R}_y) \). Moreover, due to the Lizorkin theorem\( \Gamma \) \( p(\xi) \) is a multiplier in \( L^r(\mathbb{R}^2) \). Hence \( \Gamma \) so is for \( P(\xi_1) \) in the space \( L^r(\mathbb{R}_x, L^r(\mathbb{R}_y)) = L^r(\mathbb{R}^2) \). It is not difficult to verify that \( P(\xi_1) \) depends continuously on \( \xi_1 \) with respect to the norm in \( L(L^r(\mathbb{R}_y)) \) at any point \( \xi_1 \neq 0 \). Therefore, by Theorem 3.8 of Ch. 7\( \Gamma[22] \Gamma \) we see that \( P(\xi_1) \) is also a multiplier in the space
\(L'((-k/2, k/2), L'(\mathbb{R}))_0 = L'(Q_k)_0\) considered as the space of \(k\)-periodic in \(x\) functions. The subscript \(0\) means that for functions from this space \(F_{k,x}u\) vanishes at \(\xi_1 = 0\). Since \(p(0, \xi_2) = 0\) the corresponding multiplier vanishes on \(\{u \in L'(Q_k) : F_{k,x}u = 0\ \text{ if } \xi_1 \neq 0\}\) and hence \(\Gamma\) is a bounded operator on the entire space \(L'(Q_k)\). In fact we need here an extension of that theorem for operator valued multipliers which may be discontinuous at the point \(0\). However in this case the proof presented in [22] works without any change.

Now to conclude we can use the same reiteration argument as in [6]. \(\square\)

**Remark 2** If \(f\) is of the class \(C^\infty\Gamma\) then \(u \in H^\infty(Q_k) = \cap H^s(Q_k)\).

We need also the following additional assumption:

**(N1)** \(f \in C^1(\mathbb{R})\) and, for any \(v \in L^2(\mathbb{R}^2)\) such that \(\int_{\mathbb{R}^2} f(v)v > 0\), we have

\[
\int_{\mathbb{R}^2} f(v)v < \int_{\mathbb{R}^2} f'(v)v^2
\]

and \(\int_{\mathbb{R}^2} f(tv)v > 0\ \forall t > 0\).

Calculating the derivative of \(t^{-1}\int_{\mathbb{R}^2} f(tv)v\) we see that (N1) implies (N).

Let us introduce the functional

\[L_k(v) = \int_{Q_k} \left[ \frac{1}{2} f(v)v - F(v) \right], \quad v \in X_k.\]

As we have seen \(L_k = J_k\) on \(S_k\) and \(L_k(v) \geq 0 \forall v \in X_k\).

**Lemma 4** Under Assumption (N1), \(L_k(tv)\) is a strictly increasing function of \(t > 0\), provided \(\int_{Q_k} f(v)v > 0\).

**Proof.** It follows immediately from the following elementary identity

\[
\frac{d}{dt}L_k(tv) = \frac{1}{2t} \left[ \int_{Q_k} f'(tv)t^2v - \int_{Q_k} f(tv)t^2v \right]. \quad \square
\]

We need also the following dual characterization of ground traveling waves

**Lemma 5** Suppose Assumptions 1.0–4.0 and (N1) to be satisfied. For nonzero \(u \in X_k, k \in (0, \infty)\), the following statements are equivalent:
(i) $u$ is a ground wave,

(ii) $I_k(u) = 0$ and $L_k(u) = m_k = \inf \{ L_k(v) : v \in S_k \}$,

(iii) $I_k(u) = 0 = \sup \{ I_k(v) : v \in X_k, L_k(v) = m_k \}$.

Proof. Implication (i) $\Rightarrow$ (ii) is proved in Section 2.

To prove (ii) $\Rightarrow$ (i) assume that $u \in X_k$ satisfies (ii). Since $J_k = L_k$ on $S_k \Gamma$, there exists a Lagrange multiplier $\lambda$ such that

$$\lambda I_k'(u) = J_k'(u).$$

Then

$$\lambda \langle I_k'(u), u \rangle = \langle J_k'(u), u \rangle = I_k(u) = 0.$$ 

On the other hand

$$\langle I_k'(u), u \rangle = 2\|u\|^2_k - \int_{Q_k} f'(u)u^2 - \int_{Q_k} f(u)u =$$

$$= 2I_k(u) + \int_{Q_k} f(u)u - \int_{Q_k} f'(u)u^2 =$$

$$= \int_{Q_k} f(u)u - \int_{Q_k} f'(u)u^2.$$ 

However, $\int_{Q_k} f(v)v > 0$ on $S_k \Gamma$ and $\Gamma$ due to (N1)$\Gamma \langle I_k'(u), u \rangle < 0$. Therefore, $\lambda = 0$ and $u$ is a ground wave.

Now let us prove (ii) $\Rightarrow$ (iii). For $u$ as in (ii)$\Gamma I_k(u) = 0$. Assume that there is $v \in X_k$ such that $L_k(v) = m_k$ and $I_k(v) < 0$. Then $\int_{Q_k} f(v)v > 0$ and there exists $t_0 \in (0, 1)$ such that $I_k(t_0v) = 0$. By Lemma 4$\Gamma L_k(t_0v) < L_k(v) = m_k \Gamma$ which is impossible.

Finally, we prove (iii) $\Rightarrow$ (ii). Let $u \in X_k$ satisfies (iii). Then $L_k(u) \geq m_k$. Assume that $L_k(u) > m_k$. Again we have $\int_{Q_k} f(u)u > 0$. By Lemma 4$\Gamma$ there exists $t_0 \in (0, 1)$ such that $L_k(t_0u) = m_k$. However, $I_k(t_0u) > 0$ and this contradicts (iii). □.

Now we are ready to prove the symmetry property for all kinds of ground waves we consider. As in [6] we use the approach suggested in [12] (see also [23]).

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Theorem 3 In addition to Assumptions $1^0 - 4^0$ and (N1), suppose that $f \in C^2(\mathbb{R})$. Then any ground wave $u \in X_k$, $k \in (0, \infty]$, is symmetric with respect to some line $\Delta = \{ (x, y) \in \mathbb{R}^2 : y = b \}$.

Proof. Choose $b$ in such a way that

$$\int_{\Delta^+ \cap Q_k} \left[ \frac{1}{2} f(v) v - F(v) \right] = \int_{\Delta^- \cap Q_k} \left[ \frac{1}{2} f(v) v - F(v) \right] = \frac{m_k}{2},$$

where $\Delta^+$ and $\Delta^-$ are corresponding upper and lower half-planes. Let $u^\pm$ be a symmetric (with respect to $\Delta$) function such that $u^\pm = u$ on $\Delta^\pm$. Then $u^\pm \in X_k$ and

$$L_k(u^\pm) = L_k(u) = m_k.$$ 

By Lemma 5 $I_k(u^\pm) \leq 0$. On the other hand

$$I_k(u^+) + I_k(u^-) = 2I_k(u) = 0.$$

Using Lemma 5 we conclude that $u^\pm$ is a ground wave.

To conclude that $u^\pm = u$ and hence complete the proof it is sufficient to use the unique continuation result as in [6] and just here we need the assumption $f \in C^2(\mathbb{R})$ and Lemma 3. Remark that a periodic version (with $\Pi = \Delta^\pm$) of unique continuation Theorem A.1 [6] can be proved exactly as that theorem itself. □

In addition we formulate the following direct generalization of results of [6] for decay of solitary waves.

Theorem 4 Suppose Assumptions $1^0$ and $2^0$ to be satisfied. Let $u \in X_k$, $k \in (0, \infty]$, be a traveling wave. If $k = \infty$, then

$$r^2 u \in L^\infty(\mathbb{R}^2), \quad r^2 = x^2 + y^2.$$ 

If $0 < k < \infty$, then

$$y^2 u \in L^\infty(Q_k).$$

The proof is essentially the same as in [6]. In the case $k < \infty$ one needs only to use the partially periodic Fourier transform as in Lemma 3.
4 On Nonexistence of Traveling Waves

In this section we turn to general KP equations (3) with \( \varepsilon = \pm 1 \) and discuss the nonexistence problem. We use the same approach as in [5]. However, the case of periodic waves is more involved (see the proof of Lemma 6). Here we consider traveling waves belonging to the space

\[ Y_k = \{ u \in X_k : u \in H^1(Q_k), u_{xx}, D^{-1}_{x,k}u_{yy} \in L^2(Q_k), f(u)u \in L^1(Q_k) \} \]

if \( k < \infty \) and

\[ Y = Y_\infty = \{ u \in X : u \in H^1(\mathbb{R}^2), u_{xx}, D^{-1}_{x,k}u_{yy} \in L^2_{loc}(\mathbb{R}^2), f(u)u \in L^1(\mathbb{R}^2) \} \].

First we collect some useful identities.

**Lemma 6** Suppose that \( f \) satisfies Assumptions 1\(^0\) and 2\(^0\). Let \( u \in Y_k, k \in (0, \infty] \), be a solution of equations (3). Then

\[
\int_{Q_k} \left[ c \frac{u^2}{2} + \frac{3}{2} u_x^2 + \varepsilon \frac{v^2}{2} - uf(u) + F(u) \right] = 0, \tag{14}
\]

\[
\int_{Q_k} \left[ c \frac{u^2}{2} + \frac{1}{2} u_x^2 + \varepsilon \frac{v^2}{2} + F(u) \right] = 0, \tag{15}
\]

\[
\int_{Q_k} \left[ cu^2 + u_x^2 - \varepsilon v^2 - f(u)u \right] = 0. \tag{16}
\]

**Proof.** First we remark that for any \( k \in (0, \infty) \) an extension to the case \( \varepsilon = \pm 1 \) of \( I_k(u) = 0 \) stated in Section 2. Therefore we concentrate at (14) and (15) only.

In the case of solitary waves \( (k = \infty) \) the calculations carried out in the proof of Theorem 1.1 in [5] work equally well for general nonlinearities. Therefore we look at periodic waves \( (k < \infty) \).

Fixed \( \kappa \in (0,1) \) let \( \varphi_T \in C_0^\infty(\mathbb{R}) \) be a nonnegative function such that \( \varphi_T = 1 \) on \([-T/2, T/2] \), \( \varphi_T(x) = 0 \) if \( x \geq (T + \theta) \), \( \varphi_j(x) \leq C_j/|x| \) for \( j = 1, 2, \ldots \) if \( T/2 \leq |x| \leq (T + \theta) \) (the construction of such a function will be given later on).
Multiplying the first equation (3) by \(x \varphi_T u\) and integrating over \(\mathbb{R}^2\), we get, after a number of integrations by parts:

\[
\frac{c}{2} \int \varphi_T u^2 - \int \varphi_T u f(u) + \int \varphi_T F(u) + \frac{3}{2} \int \varphi_T u_x^2 + \\
+ \varepsilon \frac{1}{2} \int \varphi_T v^2 + \frac{1}{2} \int x \varphi'_T u^2 - \int x \varphi'_T u f(u) + \int x \varphi'_T F(u) + \\
2 \int \varphi_T uu_x + \int x \varphi'_T uu_x + \frac{3}{2} \int x \varphi'_T u_x^2 + \varepsilon \frac{1}{2} \int x \varphi'_T v^2 = 0.
\]

Dividing the last identity by \(T\), we are going to pass to the limit as \(T \to 1\).

First, we point out that here the integrals containing \(\varphi'_T\) are taken over
\[Q_T \cup Q'_T \cup Q''_T = Q_T \cup \{(T/2, (T+T^\varepsilon)/2) \times \mathbb{R}\} \cup \{-(T+T^\varepsilon)/2, -T/2 \times \mathbb{R}\},
\]
while that ones containing \(\varphi'_T\) and \(\varphi''_T\) are over \(Q'_T \cup Q''_T\). Moreover, \(\varphi_T = 1\) on \(Q_T\).

Now let \(g \in L^1_{\text{loc}}(\mathbb{R}^2)\) be a function which is \(k\)-periodic in \(x\). Then it is easy to verify that

\[
\lim_{T \to \infty} \frac{1}{T} \int_{Q_T} g = \int_{Q_k} g.
\]

Next, due to the properties of \(\varphi_T\), all the integrals over \(Q'_T\) can be estimated from above by \(\int_{Q'_T} g\) with a nonnegative \(k\)-periodic in \(x\) function \(g \in L^1_{\text{loc}}(\mathbb{R}^2)\). Now

\[
\frac{1}{T} \int_{Q'_T} g \leq \frac{T^\varepsilon + 1}{T} \int_{Q_k} g.
\]

This justifies the passage to the limit and gives rise to (14).

Identity (15) can be proved exactly as (2.8) \(\Gamma[5]\) with the only change: take there cut-off functions \(\chi_j\) depending on \(y\) only.

Now we construct the function \(\varphi_T\). Fix \(\varepsilon > 0\) and let

\[
g(x) = \begin{cases} 
1 & \text{if } x \leq T/2 \\
1 - \log(x/T) & \text{if } T/2 < x \leq (T+T^\varepsilon)/2 + \varepsilon \\
0 & \text{if } x > (T+T^\varepsilon)/2 + \varepsilon
\end{cases}
\]

We choose a nonnegative function \(h \in C^\infty_0(\mathbb{R})\) such that \(\text{supp } h \subset (0, \varepsilon)\) and \(\int h = 1\) and set

\[
\tilde{\varphi}_T(x) = \int h(x-t)g(t)dt, \quad \varphi_T(x) = \tilde{\varphi}_T(|x|).
\]
For this function it is easy to verify all the properties we need. □

**Theorem 5** Suppose that \( f \in C(\mathbb{R}) \) satisfies Assumption 4’. Then there is no nontrivial traveling wave \( u \in Y_k, \ k \in (0, \infty], \) provided \( \varepsilon = +1, \) or \( \varepsilon = -1 \) and \( \mu \geq 6. \)

**Proof.** Adding (14)\( \Gamma(15) \) and subtracting (16)\( \Gamma \)we get

\[
\int_{Q_k} u_x^2 = -2\varepsilon \int_{Q_k} v^2.
\]

This rules out the case \( \varepsilon = +1. \) In the case \( \varepsilon = -1 \) (KP-I equations) the last identity together with (14) and (16)\( \Gamma \)respectively implies

\[
\int_{Q_k} \left[ c \frac{u^2}{2} + \frac{5}{2} v^2 - f(u)u + F(u) \right] = 0
\]

and

\[
\int_{Q_k} [cu^2 + 3v^2 - f(u)u] = 0.
\]

Eliminating \( v \)we get

\[
2c \int_{Q_k} u^2 = \int_{Q_k} [6 F(u) - f(u)u].
\]

If \( \mu \geq 6 \)we have

\[
2c \int_{Q_k} u^2 \leq \int_{Q_k} [\mu F(u) - f(u)u] \leq 0.
\]

Hence \( \Gamma u = 0 \) and we conclude. □

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