# A counterexample to an integer analogue of Carathéodory's theorem

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**Abstract**

For $n \geq 6$ we provide a counterexample to the conjecture that every integral vector of a $n$-dimensional integral polyhedral pointed cone $C$ can be written as a nonnegative integral combination of at most $n$ elements of the Hilbert basis of $C$. In fact, we show that in general at least $\lfloor 7/6 \cdot n \rfloor$ elements of the Hilbert basis are needed.

**Keywords:** Integral pointed cones, Hilbert basis, integral Carathéodory property.

## 1 Introduction

Throughout this paper we resort to the following notation. For integral points $z^1, \dots, z^k \in \mathbb{Z}^n$, the set

$$C = \mathrm{pos}\{z^1, \dots, z^k\} = \left\{ \sum_{i=1}^{k} \lambda_i z^i \,:\, \lambda \in \mathbb{R}^k_{\geq 0} \right\}$$

is called an *integral polyhedral cone* generated by $\{z^1, \dots, z^k\}$. It is called *pointed* if the origin is a vertex of $C$ and it is called *unimodular* if the set of generators $\{z^1, \dots, z^k\}$ of $C$ forms part of a basis of the lattice $\mathbb{Z}^n$. By Gordan's lemma [G1873] the semigroup $C \cap \mathbb{Z}^n$ is finitely generated for any integral polyhedral cone $C$, i.e., there exist finitely many vectors $h^1, \dots, h^m$ such that every $z \in C \cap \mathbb{Z}^n$ has a representation of the form $z = \sum_{i=1}^{m} m_i h^i$, $m_i \in \mathbb{Z}_{\geq 0}$. It was pointed out by van der Corput [Cor31] that for a pointed integral polyhedral cone $C$ there exists a uniquely determined minimal (w.r.t. inclusions) finite generating system $\mathcal{H}(C)$ of $C \cap \mathbb{Z}^n$ which may be characterized as the set of all irreducible integral vectors contained in $C$. More precisely,

$$\mathcal{H}(C) = \Big\{ z \in C \cap \mathbb{Z}^n \backslash \{0\} : z \text{ cannot be written as the sum of two other elements of } C \cap \mathbb{Z}^n \backslash \{0\} \Big\}. \tag{1.1}$$

The set $\mathcal{H}(C)$ is usually called the *Hilbert basis* of $C$. Although Hilbert bases play a role in various fields of mathematics, like *combinatorial convexity and*

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*toric varieties* (cf. e.g. [DHH98], [Ewa96], [Oda88], [Stu96]), *polynomial rings and ideals* (cf. e.g. [BG98], [BGT97]) or in *integer programming* (cf. e.g. [Gra75], [GP79], [Sch80], [Seb90],[Wei98]), their structure is not very well understood yet. A first systematic study was given by Sebö [Seb90]. In particular, the following three conjectures about the "nice" geometrical structure of Hilbert bases of an integral pointed polyhedral cone $C \subset \mathbb{R}^n$ are due to him:

**(**Unimodular **H**ilbert **P**artitioning**)** *There exist unimodular cones* $C_i$, $i \in I$, *generated by elements of* $\mathcal{H}(C)$ *such that* i) $C = \cup_{i \in I} C_i$ *and* ii) $C_i \cap C_j$ *is a face both of* $C_i$ *and* $C_j$, $i, j \in I$.

**(**Unimodular **H**ilbert **C**over**)** *There exist unimodular cones* $C_i$, $i \in I$, *generated by elements of* $\mathcal{H}(C)$ *such that* $C = \cup_{i \in I} C_i$.

**(**Integral **C**arathéodory **P**roperty**)** *Each integral vector* $z \in C$ *can be written as a nonnegative integral combination of at most* $n$ *elements of* $\mathcal{H}(C)$.

Let us remark that the question whether $n$ elements of the Hilbert basis are sufficient to express any integral vector of the cone as a nonnegative integral combination (and thus having a nice counterpart to Carathéodory's theorem) has already been raised by Cook, Fonlupt&Schrijver [CFS86].

Obviously, (UHP) implies (UHC) and (UHC) implies (ICP). Sebö also verified (UHP) (and thus all three conjectures) in dimensions $n \leq 3$ [Seb90]. An independent proof was given by Aguzzoli&Mundici [AM94] in the context of desingularization of 3-dimensional toric varieties. However, in dimensions $n \geq 4$ (UHP) does not hold anymore as it was shown by Bouvier&Gonzalez-Sprinberg [BGS92]. In order to attack algorithmically the (UHC)-conjecture Firla&Ziegler [FZ97] introduced the notation of a *binary unimodular Hilbert covering* which is a stronger property than (UHC) but weaker than (UHP) and they falsified this property in dimensions $n \geq 5$.

Recently, Bruns&Gubeladze [BG98] managed to give a counterexample to the original (UHC)-conjecture in dimensions $n \geq 6$. We show in this note that also the weakest of the three conjectures, the (ICP)-conjecture, does not hold in dimensions $n \geq 6$. To this end we define for a pointed integral polyhedral cone $C \subset \mathbb{R}^n$ its *Carathéodory rank* (as in [BG98]) by

$$\mathrm{CR}(C) = \max_{z \in C \cap \mathbb{Z}^n} \min\{m : z = n_1 h^1 + \cdots + n_m h^m,\, n_i \in \mathbb{N},\ h^i \in \mathcal{H}(C)\},$$

and moreover, let

$$\mathrm{h}(n) = \max\{\mathrm{CR}(C) : C \subset \mathbb{R}^n \text{ an integral pointed polyhedral cone}\}$$

be the *maximal Carathéodory rank* in dimension $n$. With this notation the (ICP)-conjecture claims $\mathrm{CR}(C) \leq n$, or equivalent, $\mathrm{h}(n) = n$ which holds in dimensions $n \leq 3$. A first general upper bound on $\mathrm{h}(n)$ was given by Cook, Fonlupt&Schrijver [CFS86]. They proved $\mathrm{h}(n) \leq 2n - 1$ and they also verified the (ICP)-conjecture for certain cones arising from perfect graphs. Another class of cones satisfying (ICP) is described in [HW97]. The bound $2n - 1$ was improved by Sebö [Seb90] to $\mathrm{h}(n) \leq 2n - 2$ which is currently the best known estimate. Moreover it is known that "almost" every integral vector of a cone can be written as an integral combination of at most $2n - 3$, $n \geq 3$, elements of its Hilbert basis [BG98]. Here we prove the following lower bound

**Theorem 1.1.**

$$\mathrm{h}(n) \geq \left\lfloor \frac{7}{6} n \right\rfloor,$$

*where* $\lfloor x \rfloor$ *denotes the largest integer not greater than* $x$, $x \in \mathbb{R}$.

Of course, this result implies that (ICP) is false in dimensions $n \geq 6$. Moreover, Theorem 1.1 shows that there is no universal constant $c$ such that any integral vector can be represented as a nonnegative linear combination of at most $n + c$ elements of the Hilbert basis.

The proof of Theorem 1.1 consists basically of a 6-dimensional cone $C_6$ with $\mathrm{CR}(\mathrm{C}_6) = 7$. This cone is, up to a different embedding, the same cone already used by Bruns&Gubeladze for disproving (UHC) and it will be described in the next section. (Note that (ICP) and the stronger properties mentioned are invariant under unimodular integral linear transformations.) Using this cone $C_6$ the proof of Theorem 1.1 runs as follows:

*Proof.* First, we assume $n = 6 \cdot p$, $p \in \mathbb{N}$, and we show inductively w.r.t. $p$ that there exist $(6 \cdot p)$-dimensional cones $C_{6 \cdot p}$ with $\mathrm{CR}(C_{6 \cdot p}) = 7 \cdot p$. For $p = 1$ we have the cone $C_6$ and therefore, let $p > 1$. Now we embed the cone $C_{6 \cdot (p-1)}$ and $C_6$ into two pairwise orthogonal lattice subspaces of $\mathbb{R}^{6 \cdot p}$ and we denote these embeddings by $\tilde{C}_{6 \cdot (p-1)}$ and $\tilde{C}_6$, respectively. With $C_{6 \cdot p} = \tilde{C}_{6 \cdot (p-1)} \times \tilde{C}_6 = \mathrm{pos}\{\tilde{C}_{6 \cdot (p-1)}, \tilde{C}_6\}$ it is quite easy to see that $\mathrm{CR}(C_{6 \cdot p}) = \mathrm{CR}(\tilde{C}_{6 \cdot (p-1)}) + \mathrm{CR}(\tilde{C}_6) = \mathrm{CR}(C_{6 \cdot (p-1)}) + \mathrm{CR}(C_6) = 7 \cdot p$.

For the remaining dimensions $n = 6 \cdot (p-1) + r$, $p \geq 1$, $r \in \{1, \ldots, 5\}$, we apply the same construction, but instead of $\tilde{C}_6$ we supplement $\tilde{C}_{6 \cdot (p-1)}$ by an arbitrary $r$-dimensional cone. □

## 2 The counterexample $C_6$ to (ICP)

The cone $C_6$ is generated by the following 10 integral vectors $z^1, \ldots, z^{10}$

$$\begin{aligned}
z^1 &= (0,\,1,\,0,\,0,\,0,\,0)^\intercal, & z^6 &= (1,\,0,\,2,\,1,\,1,\,2)^\intercal,\\
z^2 &= (0,\,0,\,1,\,0,\,0,\,0)^\intercal, & z^7 &= (1,\,2,\,0,\,2,\,1,\,1)^\intercal,\\
z^3 &= (0,\,0,\,0,\,1,\,0,\,0)^\intercal, & z^8 &= (1,\,1,\,2,\,0,\,2,\,1)^\intercal,\\
z^4 &= (0,\,0,\,0,\,0,\,1,\,0)^\intercal, & z^9 &= (1,\,1,\,1,\,2,\,0,\,2)^\intercal,\\
z^5 &= (0,\,0,\,0,\,0,\,0,\,1)^\intercal, & z^{10} &= (1,\,2,\,1,\,1,\,2,\,0)^\intercal.
\end{aligned}$$

Observe that all the generators are contained in the hyperplane $\{x \in \mathbb{R}^6 : ax = 1\}$ where $a = (-5, 1, 1, 1, 1, 1)$. Moreover, $C_6$ has 27 facets, 22 of them are simplicial, i.e., generated by five vectors. The remaining five facets are generated by six vectors and can be described as cones over 4-dimensional polytopes which are bipyramids over 3-dimensional simplices.

The first important property of $C_6$ is that the set of generators coincides with the Hilbert basis, i.e.,

$$\mathcal{H}(C_6) = \{z^1, \ldots, z^{10}\}. \tag{2.1}$$

On account of (1.1), there is a straightforward way to check (2.1) by computing all integral vectors in the zonotope $Z = \{x \in \mathbb{R}^6 : x = \sum_{i=1}^{10} \lambda_i z^i,\ 0 \leq \lambda_i \leq 1\}$

and then to verify that for each such nontrivial integral vector $w$ there exists a $z^i$, $1 \le i \le 10$, such that $w - z^i \in C$. However, there are also computer programs available which have routines for computing the Hilbert basis of an integral polyhedral cone (see e.g. `normaliz` [BK] or `bastat` [Pot96]).

Let $S_6$ be the semigroup $C_6 \cap \mathbb{Z}^6$. The automorphism group $\operatorname{Aut}(S_6)$ of $S_6$ is surprisingly large. In fact, the following permutations of the generators

$$\sigma = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10) \quad \text{and} \quad \tau = (2\ 5)(3\ 4)(7\ 10)(8\ 9)$$

induce automorphisms of $S_6$ (the number $i$ stands for the generator $z^i$). The group generated by them is isomorphic to the dihedral group $D_{10}$ of order 10, and coincides with the subgroup of $\operatorname{Aut}(S_6)$ stabilizing one (or both) of the subsets $F_1 = \{z^1, \dots, z^5\}$ and $F_2 = \{z^6, \dots, z^{10}\}$ ($F_1$ and $F_2$ generate facets of $C_6$). But there are also automorphisms exchanging $F_1$ and $F_2$, for example

$$\rho = (1\ 6)(2\ 8\ 5\ 9)(3\ 10\ 4\ 7);$$

$\sigma$ and $\rho$ generate $\operatorname{Aut}(S_6)$. Since $\tau = \rho^2$ and $\rho\sigma\rho^{-1} = \sigma^2$, $\operatorname{Aut}(S_6)$ is isomorphic to a semidirect product of $\mathbb{Z}/(5)$ and $\mathbb{Z}/(4)$; in particular $\#\operatorname{Aut}(S_6) = 20$. Moreover, $\operatorname{Aut}(S_6)$ operates transitively on $\mathcal{H}(C_6)$. In order to verify these claims one examines the embedding of $C_6$ into $\mathbb{R}^{27}$ by the primitive integral linear forms defining the support facets of $C_6$; that $\sigma$ and $\tau$ induce automorphisms is already visible in the definition of $C_6$.

Let

$$g = (9, 13, 13, 13, 13, 13)^{\mathsf{T}} = z^1 + 3z^2 + 5z^4 + 2z^5 + z^8 + 5z^9 + 3z^{10} \in C_6. \tag{2.2}$$

We verified that $g$ can not be written as a nonnegative integral combination of at most 6 elements of $\mathcal{H}(C_6)$ by two different methods. In [BG98] it has been shown that if (ICP) would hold for $C_6$, then every integral vector could even be written as the nonnegative integral combination of at most 6 *linearly independent* elements of the Hilbert basis. Hence all what one has to do is to solve all linear systems of the form $(z^{k_1}, \dots, z^{k_6})x = g$ for any choice $\{k_1, \dots, k_6\} \subset \{1, \dots, 10\}$ such that $z^{k_1}, \dots, z^{k_6}$ are linearly independent and then to check that no integral nonnegative solution occurs. The second approach verifies the claim via an integer linear program which reads as follows:

$$\text{minimize} \quad \sum_{i=1}^{10} u_i \qquad \text{subject to} \qquad \sum_{i=1}^{10} n_i z^i = g,$$
$$0 \le n_i \le 13 \cdot u_i, \quad n_i \in \mathbb{Z}, \qquad u_i \in \{0,1\}, \quad 1 \le i \le 10.$$

The $0/1$-variables $u_i$ control which element is used for the representation of the vector $g$. If $z^k$ is used, i.e., $u_k = 1$, then the scalar $n_k$ in front of $z^k$ can not exceed the maximal entry in $g$, because all vectors are nonnegative and we are just looking for nonnegative representations of $g$. Hence a solution of the above integer linear program gives a representation of $g$ with a minimal number of vectors of the Hilbert basis. In order to solve this program we used the program `SIP` (cf. [MW98], [M98]) and the output is the representation of $g$ in (2.2) with 7 elements of the Hilbert basis.

If we use the function $\mathrm{dg}(x) = ax$ as a graduation on the set $C_6 \cap \mathbb{Z}^6$ then one can show that, among all lattice points in $C_6$ violating (ICP), the two lattice points $g$ (cf.(2.2)) and $h = (11, 15, 15, 15, 15, 15)^\intercal \in C_6$ are the only ones with lowest degree ($\mathrm{dg}(g) = \mathrm{dg}(h) = 20$). (The automorphisms $\sigma$ and $\tau$ leave $g$ invariant whereas $\rho(g) = h$.)

By the way, the lowest degree lattice point violating (UHC) is

$$t = (5, 7, 7, 7, 7, 7)^\intercal = z^1 + \cdots + z^{10};$$

it has degree 10 and is obviously invariant under the full automorphism group.

Finally, we want to show $\mathrm{CR}(C_6) = 7$. Of course, the above argumentation yields $\mathrm{CR}(C_6) \geq 7$. Let

$$\begin{aligned}
K_1 &= \mathrm{pos}\{z^3, z^4, z^5, z^6, z^7, z^8\}, & K_2 &= \mathrm{pos}\{z^2, z^3, z^5, z^6, z^7, z^8\}, \\
K_3 &= \mathrm{pos}\{z^2, z^3, z^5, z^6, z^7, z^9\}, & K_4 &= \mathrm{pos}\{z^1, z^2, z^5, z^7, z^8, z^{10}\}, \\
K_5 &= \mathrm{pos}\{z^2, z^3, z^4, z^5, z^7, z^8\}, & K_6 &= \mathrm{pos}\{z^2, z^3, z^4, z^7, z^8, z^{10}\}, \\
K_7 &= \mathrm{pos}\{z^1, z^2, z^3, z^4, z^5, z^7\}, & K_8 &= \mathrm{pos}\{z^1, z^2, z^3, z^4, z^7, z^{10}\}, \\
K_9 &= \mathrm{pos}\{z^1, z^2, z^5, z^6, z^7, z^8\}, & K_{10} &= \mathrm{pos}\{z^3, z^4, z^6, z^7, z^8, z^{10}\}, \\
K_{11} &= \mathrm{pos}\{z^2, z^4, z^5, z^7, z^8, z^{10}\}, & K_{12} &= \mathrm{pos}\{z^2, z^3, z^4, z^5, z^6, z^8\}, \\
K_{13} &= \mathrm{pos}\{z^2, z^3, z^6, z^7, z^8, z^{10}\}, & K_{14} &= \mathrm{pos}\{z^1, z^2, z^4, z^5, z^7, z^{10}\}, \\
K_{15} &= \mathrm{pos}\{z^1, z^2, z^7, z^8, z^9, z^{10}\}, & K_{16} &= \mathrm{pos}\{z^1, z^2, z^6, z^7, z^8, z^9\}, \\
K_{17} &= \mathrm{pos}\{z^2, z^6, z^7, z^8, z^9, z^{10}\}, & K_{18} &= \mathrm{pos}\{z^1, z^2, z^3, z^5, z^7, z^9\}, \\
K_{19} &= \mathrm{pos}\{z^1, z^2, z^5, z^6, z^7, z^9\}, & K_{20} &= \mathrm{pos}\{z^1, z^5, z^6, z^7, z^8, z^9\}, \\
K_{21} &= \mathrm{pos}\{z^1, z^2, z^4, z^5, z^8, z^{10}\}, & K_{22} &= \mathrm{pos}\{z^2, z^3, z^6, z^7, z^9, z^{10}\}.
\end{aligned}$$

Via a computer program (for details see [BG98]) we have checked that the cones $K_i$ cover $C_6$, i.e., $C_6 = \cup_{i=1}^{22} K_i$. Except for the first three cones, all these simplicial cones are unimodular and each sublattice spanned by the generators of $K_i$, $1 \leq i \leq 3$, has index 2 w.r.t. $\mathbb{Z}^6$. It turns out that the Hilbert bases of the cones $K_1, K_2, K_3$ consist of the generators and the additional point

$$u = (1, 1, 1, 2, 1, 2)^\intercal = z^4 + z^9 = \frac{1}{2}\left(z^3 + z^5 + z^6 + z^7\right).$$

The last relation shows that $u$ is contained in the 4-face $\mathrm{pos}\{z^3, z^5, z^6, z^7\}$ of each of the cones $K_i$, $1 \leq i \leq 3$. Thus with the help of $u$ we can subdivide each of these cones into 4 unimodular cones and $\mathrm{CR}(C) = 7$ is established.

The embedding of $C_6$ given above has been chosen because it displays many automorphisms of $S_6 = C_6 \cap \mathbb{Z}^6$. However, one can also give an embedding by 0/1-vectors: set

$$\begin{aligned}
y^1 &= (0, 1, 1, 0, 0, 0)^\intercal, & y^2 &= (0, 1, 1, 1, 0, 0)^\intercal, \\
y^3 &= (0, 1, 0, 1, 1, 0)^\intercal, & y^4 &= (0, 1, 0, 0, 1, 1)^\intercal, \\
y^5 &= (0, 1, 0, 0, 0, 1)^\intercal, & y^6 &= (1, 0, 0, 1, 0, 1)^\intercal, \\
y^7 &= (1, 0, 0, 0, 1, 0)^\intercal, & y^8 &= (1, 0, 1, 0, 0, 1)^\intercal, \\
y^9 &= (1, 0, 0, 1, 0, 0)^\intercal, & y^{10} &= (1, 0, 1, 0, 1, 0)^\intercal;
\end{aligned}$$

then there is a unimodular integral linear transformation $\phi : \mathbb{Z}^6 \to \mathbb{Z}^6$ such that $\phi(z_i) = y_i$ for $i = 1, \ldots, 10$. Hence (ICP) (and the stronger properties) do not even hold in the class of cones generated by 0/1-vectors. The vector $g$ above that disproves (ICP) is transformed into $(9, 11, 8, 8, 8, 8)^\intercal$. (We are grateful to T. Hibi and A. Sebö for asking us about the existence of a 0/1-embedding.)

# 3 Remarks

Finally, we want to remark that the name *Hilbert basis* was introduced by Giles&Pulleyblank in their investigations of so called TDI-systems in integer linear programming [GP79]. An integral linear system $Ax \leq b$, $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, is called TDI (totally dual integral) if the minimum in the linear programming duality equation

$$\min\{b^\intercal y : A^\intercal y = c, \quad y \geq 0\} = \max\{c^\intercal x : Ax \leq b\} \tag{3.1}$$

can be achieved by an integer vector $y \in \mathbb{Z}^m$ for *each* integer vector $c \in \mathbb{Z}^n$ for which the optima exist. In this context the (ICP)-conjecture has the following interpretation (for details we refer to [GP79] and [CFS86]): *Let $Ax \leq b$ be a TDI-system, such that the polyhedron $\{x \in \mathbb{R}^n : Ax \leq b\}$ is of dimension $n$. Let $c \in \mathbb{Z}^n$ such that the minimum in* (3.1) *exists. Then the minimum can be achieved by an integral vector $y \in \mathbb{Z}^m$ with at most $n$ nonzero variables.*

However, each integral pointed cone $C \subset \mathbb{R}^n$ gives rise to a TDI-system $Ax \leq b$ and an integral vector $c \in \mathbb{Z}^n$ such that the minimum in (3.1) exists and this minimum can only be achieved by an integral vector with $\mathrm{CR}(C)$ nonzero variables. To see this just set $b = 0$, let $A$ be the matrix with rows consisting of the Hilbert basis of $C$ and for $c$ we can choose any lattice point in $C$ which can only be written as nonnegative integral combination of $\mathrm{CR}(C)$ elements of the Hilbert basis. Hence Theorem 1.1 and the construction used in its proof leads to

**Corollary 3.1.** *Let $Ax \leq b$, $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, be a TDI-system and let $c \in \mathbb{Z}^n$ such that the minimum in* (3.1) *exists. In general the minimum cannot be achieved by an integral vector $y \in \mathbb{Z}^m$ with less than $\lfloor (7/6)n \rfloor$ nonzero variables.*

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