A counterexample to an integer analogue of Carathéodory's theorem

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Abstract

For $n \geq 6$ we provide a counterexample to the conjecture that every integral vector of a n-dimensional integral polyhedral pointed cone C can be written as a nonnegative integral combination of at most n elements of the Hilbert basis of C. In fact, we show that in general at least $\lfloor 7/6 \cdot n \rfloor$ elements of the Hilbert basis are needed.

Keywords: Integral pointed cones, Hilbert basis, integral Carathéodory property.

1 Introduction

Throughout this paper we resort to the following notation. For integral points $z^1, \dots, z^k \in \mathbb{Z}^n$, the set

$$C = \operatorname{pos}\{z^1, \dots, z^k\} = \left\{ \sum_{i=1}^{k} \lambda_i z^i \,:\, \lambda \in \mathbb{R}^k_{\geq 0} \right\}$$

is called an *integral polyhedral cone* generated by $\{z^1, \dots, z^k\}$. It is called *pointed* if the origin is a vertex of C and it is called *unimodular* if the set of generators $\{z^1, \dots, z^k\}$ of C forms part of a basis of the lattice $\mathbb{Z}^n$. By Gordan's lemma [G1873] the semigroup $C \cap \mathbb{Z}^n$ is finitely generated for any integral polyhedral cone C, i.e., there exist finitely many vectors $h^1, \dots, h^m$ such that every $z \in C \cap \mathbb{Z}^n$ has a representation of the form $z = \sum_{i=1}^{m} m_i h^i$, $m_i \in \mathbb{Z}_{\geq 0}$. It was pointed out by van der Corput [Cor31] that for a pointed integral polyhedral cone C there exists a uniquely determined minimal (w.r.t. inclusions) finite generating system $\mathcal{H}(C)$ of $C \cap \mathbb{Z}^n$ which may be characterized as the set of all irreducible integral vectors contained in C. More precisely,

$$\mathcal{H}(C) = \Big\{ z \in C \cap \mathbb{Z}^n \backslash \{0\} : z \text{ cannot be written as the sum of two other elements of } C \cap \mathbb{Z}^n \backslash \{0\} \Big\}. \tag{1.1}$$

The set $\mathcal{H}(C)$ is usually called the *Hilbert basis* of C. Although Hilbert bases play a role in various fields of mathematics, like *combinatorial convexity and*

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toric varieties (cf. e.g. [DHH98], [Ewa96], [Oda88], [Stu96]), *polynomial rings and ideals* (cf. e.g. [BG98], [BGT97]) or in *integer programming* (cf. e.g. [Gra75], [GP79], [Sch80], [Seb90],[Wei98]), their structure is not very well understood yet. A first systematic study was given by Sebö [Seb90]. In particular, the following three conjectures about the "nice" geometrical structure of Hilbert bases of an integral pointed polyhedral cone $C \subset \mathbb{R}^n$ are due to him:

(**U**nimodular **H**ilbert **P**artitioning) *There exist unimodular cones C_i, $i \in I$, generated by elements of $\mathcal{H}(C)$ such that* i) $C = \cup_{i \in I} C_i$ *and* ii) $C_i \cap C_j$ *is a face both of C_i and C_j, $i, j \in I$.*

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(**I**ntegral **C**arathéodory **P**roperty) *Each integral vector $z \in C$ can be written as a nonnegative integral combination of at most n elements of $\mathcal{H}(C)$.*

Let us remark that the question whether n elements of the Hilbert basis are sufficient to express any integral vector of the cone as a nonnegative integral combination (and thus having a nice counterpart to Carathéodory's theorem) has already been raised by Cook, Fonlupt&Schrijver [CFS86].

Obviously, (UHP) implies (UHC) and (UHC) implies (ICP). Sebö also verified (UHP) (and thus all three conjectures) in dimensions $n \leq 3$ [Seb90]. An independent proof was given by Aguzzoli&Mundici [AM94] in the context of desingularization of 3-dimensional toric varieties. However, in dimensions $n \geq 4$ (UHP) does not hold anymore as it was shown by Bouvier&Gonzalez-Sprinberg [BGS92]. In order to attack algorithmically the (UHC)-conjecture Firla&Ziegler [FZ97] introduced the notation of a *binary unimodular Hilbert covering* which is a stronger property than (UHC) but weaker than (UHP) and they falsified this property in dimensions $n \geq 5$.

Recently, Bruns&Gubeladze [BG98] managed to give a counterexample to the original (UHC)-conjecture in dimensions $n \geq 6$. We show in this note that also the weakest of the three conjectures, the (ICP)-conjecture, does not hold in dimensions $n \geq 6$. To this end we define for a pointed integral polyhedral cone $C \subset \mathbb{R}^n$ its *Carathéodory rank* (as in [BG98]) by

$$\mathrm{CR}(C) = \max_{z \in C \cap \mathbb{Z}^n} \min\{m : z = n_1 h^1 + \cdots + n_m h^m,\, n_i \in \mathbb{N},\ h^i \in \mathcal{H}(C)\},$$

and moreover, let

$$\mathrm{h}(n) = \max\{\mathrm{CR}(C) : C \subset \mathbb{R}^n \text{ an integral pointed polyhedral cone}\}$$

be the *maximal Carathéodory rank* in dimension n. With this notation the (ICP)-conjecture claims $\mathrm{CR}(C) \leq n$, or equivalent, $\mathrm{h}(n) = n$ which holds in dimensions $n \leq 3$. A first general upper bound on $\mathrm{h}(n)$ was given by Cook, Fonlupt&Schrijver [CFS86]. They proved $\mathrm{h}(n) \leq 2n-1$ and they also verified the (ICP)-conjecture for certain cones arising from perfect graphs. Another class of cones satisfying (ICP) is described in [HW97]. The bound $2n-1$ was improved by Sebö [Seb90] to $\mathrm{h}(n) \leq 2n-2$ which is currently the best known estimate. Moreover it is known that "almost" every integral vector of a cone can be written as an integral combination of at most $2n-3$, $n \geq 3$, elements of its Hilbert basis [BG98]. Here we prove the following lower bound

Theorem 1.1.

$$\mathrm{h}(n) \geq \left\lfloor \frac{7}{6} n \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x, $x \in \mathbb{R}$.

Of course, this result implies that (ICP) is false in dimensions $n \geq 6$. Moreover, Theorem 1.1 shows that there is no universal constant c such that any integral vector can be represented as a nonnegative linear combination of at most $n + c$ elements of the Hilbert basis.

The proof of Theorem 1.1 consists basically of a 6-dimensional cone C_6 with $\mathrm{CR}(\mathrm{C}_6) = 7$. This cone is, up to a different embedding, the same cone already used by Bruns&Gubeladze for disproving (UHC) and it will be described in the next section. (Note that (ICP) and the stronger properties mentioned are invariant under unimodular integral linear transformations.) Using this cone C_6 the proof of Theorem 1.1 runs as follows:

Proof. First, we assume $n = 6 \cdot p$, $p \in \mathbb{N}$, and we show inductively w.r.t. p that there exist $(6 \cdot p)$-dimensional cones $C_{6 \cdot p}$ with $\mathrm{CR}(C_{6 \cdot p}) = 7 \cdot p$. For $p = 1$ we have the cone C_6 and therefore, let $p > 1$. Now we embed the cone $C_{6 \cdot (p-1)}$ and C_6 into two pairwise orthogonal lattice subspaces of $\mathbb{R}^{6 \cdot p}$ and we denote these embeddings by $\tilde{C}_{6 \cdot (p-1)}$ and $\tilde{C}_6$, respectively. With $C_{6 \cdot p} = \tilde{C}_{6 \cdot (p-1)} \times \tilde{C}_6 = \mathrm{pos}\{\tilde{C}_{6 \cdot (p-1)}, \tilde{C}_6\}$ it is quite easy to see that $\mathrm{CR}(C_{6 \cdot p}) = \mathrm{CR}(\tilde{C}_{6 \cdot (p-1)}) + \mathrm{CR}(\tilde{C}_6) = \mathrm{CR}(C_{6 \cdot (p-1)}) + \mathrm{CR}(C_6) = 7 \cdot p$.

For the remaining dimensions $n = 6 \cdot (p-1) + r$, $p \geq 1$, $r \in \{1, \ldots, 5\}$, we apply the same construction, but instead of $\tilde{C}_6$ we supplement $\tilde{C}_{6 \cdot (p-1)}$ by an arbitrary r-dimensional cone. □

2 The counterexample C_6 to (ICP)

The cone C_6 is generated by the following 10 integral vectors $z^1, \ldots, z^{10}$

$$\begin{aligned}
z^1 &= (0, 1, 0, 0, 0, 0)^\intercal, & z^6 &= (1, 0, 2, 1, 1, 2)^\intercal, \\
z^2 &= (0, 0, 1, 0, 0, 0)^\intercal, & z^7 &= (1, 2, 0, 2, 1, 1)^\intercal, \\
z^3 &= (0, 0, 0, 1, 0, 0)^\intercal, & z^8 &= (1, 1, 2, 0, 2, 1)^\intercal, \\
z^4 &= (0, 0, 0, 0, 1, 0)^\intercal, & z^9 &= (1, 1, 1, 2, 0, 2)^\intercal, \\
z^5 &= (0, 0, 0, 0, 0, 1)^\intercal, & z^{10} &= (1, 2, 1, 1, 2, 0)^\intercal.
\end{aligned}$$

Observe that all the generators are contained in the hyperplane $\{x \in \mathbb{R}^6 : ax = 1\}$ where $a = (-5, 1, 1, 1, 1, 1)$. Moreover, C_6 has 27 facets, 22 of them are simplicial, i.e., generated by five vectors. The remaining five facets are generated by six vectors and can be described as cones over 4-dimensional polytopes which are bipyramids over 3-dimensional simplices.

The first important property of C_6 is that the set of generators coincides with the Hilbert basis, i.e.,

$$\mathcal{H}(C_6) = \{z^1, \ldots, z^{10}\}. \tag{2.1}$$

On account of (1.1), there is a straightforward way to check (2.1) by computing all integral vectors in the zonotope $Z = \{x \in \mathbb{R}^6 : x = \sum_{i=1}^{10} \lambda_i z^i,\ 0 \leq \lambda_i \leq 1\}$