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How to Cut a Cake almost Fairly

ZIB-Report 02-23 (May 2002)

HOW TO CUT A CAKE ALMOST FAIRLY

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ABSTRACT. In the cake cutting problem, $n \geq 2$ players want to cut a cake into n pieces so that every player gets a "fair" share of the cake by his own measure. We describe a protocol with $n-1$ cuts in which each player can enforce to get a share of at least $1/(2n-2)$. Moreover we show that no protocol with $n-1$ cuts can guarantee a better fraction.

1. INTRODUCTION

Bob and Carol own a cake which they want to split into two parts to be allotted between them. Carol likes the right side of the cake since it is thicker with frosting than the left side. Bob is on a diet and does not care much about the frosting, but still he would like the cherry in the middle of the cake. Moreover Bob likes the nuts, but they are scattered with concentrations on both the left and right sides, but not many in the middle. Is there any protocol which enables Bob and Carol to cut the cake into two pieces such that both will get at least half of the cake by their own measure? The answer to this question is "Yes", and there is a quite simple solution due to Steinhaus [3] from 1948: Bob cuts the cake into two pieces, and Carol chooses her piece out of the two. Bob is sure to get at least half the cake if he cuts the cake into two equal pieces by his measure. Carol is sure to get at least half the cake by her measure by choosing the better half.

In a more general and more mathematical formulation, there are n *players* $1, \ldots, n$ and a cake C. Every player p ($1 \leq p \leq n$) has his own measure μ_p on the subsets of C. These measures satisfy $\mu_p(X) \geq 0$ for all $X \subseteq C$, and $\mu_p(X)+\mu_p(X') = \mu_p(X \cup X')$ for all disjoint subsets $X, X' \subseteq C$. For every $X \subseteq C$ and for every λ with $0 \leq \lambda \leq 1$, there exists a piece $X' \subseteq X$ such that $\mu_p(X') = \lambda \cdot \mu_p(X)$.

A *protocol* is a step by step interactive procedure that can issue queries to the players whose answers may affect future decisions. Feasible queries are: "Cut the second

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piece of cake into eight subpieces" or "Choose your favorite three pieces out of these twenty pieces" or "The third piece is alloted to the first player". The protocol has no information on the measures μ_p of the players – this is private information. The protocol cannot ask two players to jointly cut a piece into subpieces; every cut is to be done by a single player in complete isolation, and without the interaction of other players. Moreover, if all players obey the protocol then each participant will end up with a piece after finitely many steps. A *strategy* of a player is an adaptive sequence of moves consistent with the protocol. For a real number β with $0 \leq \beta \leq 1$, a β-*strategy* of a player is a strategy that will guarantee him at least a fraction β of the cake according to his own measure, independently of the play of the other $n-1$ players. A protocol is called β-*fair*, if every player has a β-strategy. A protocol for n players is called *perfectly fair*, if every player has a $\frac{1}{n}$-strategy. A protocol for n players is called *frugal*, if it only uses $n-1$ cuts; note that no protocol can do with a smaller number of cuts, since n pieces are to be produced.

Even & Paz [2] show that for $n \geq 3$ players, there does not exist a perfectly fair protocol that does only $n-1$ cuts. Moreover, [2] describe a perfectly fair protocol for $n \geq 3$ players that uses only $n \log_2 n$ cuts. Tighter results are known for small values of n: For $n = 2$ players, the Steinhaus protocol yields a perfectly fair protocol with a single cut. For $n = 3$ and $n = 4$ players, Even & Paz [2] presents perfectly fair protocols that make at most n cuts. Webb [4] presents a perfectly fair protocol for $n = 5$ players with 6 cuts, and he shows that no perfectly fair protocol exists that uses only 5 cuts. For more information on this problem and on other variants, we refer the reader to the book by Brams & Taylor [1].

In this paper we are interested in frugal protocols for n players. What is the largest value β, for which there exists a β-fair frugal protocol for $n = 3$ players? The result of Even & Paz [2] mentioned above implies that $\beta < \frac{1}{3}$. We will show that in fact the best possible value for β is $\frac{1}{4}$. This is a special case of our main theorem:

Theorem 1.1. *For $n \geq 2$ players, there exists a $1/(2n-2)$-fair frugal protocol for cake cutting. Moreover, there does not exist a β-fair frugal protocol for cake cutting with $\beta > 1/(2n-2)$.*

A consequence of our main theorem is that using n or more cuts in a protocol can buy us at most a factor of 2 in the fairness.

2. Proof of the upper bound

In this section, we prove the positive statement in Theorem 1.1. We now define a recursive protocol that in certain terminal steps will have to divide a single piece among a single player. In such a case with $n = 1$, the whole cake C is alloted to this player. For $n \geq 2$ players, we use the following protocol.

- **(S1):** The first player cuts the cake into a left piece C_L and a right piece C_R.
- **(S2):** Set $x_1 = 1$. For $p = 2, \ldots, n$ the player p chooses an integer x_p with $0 \leq x_p \leq n$.
- **(S3):** The players are divided into two non-empty groups L and R, such that $x_p \geq |L|$ holds for every player $p \in L$ and such that $x_p \leq |L|$ holds for every player $p \in R$.

(S4): The players in L recursively share the left piece C_L. The players in R recursively share the right piece C_R.

We first argue that step (S3) of the protocol indeed can be implemented.

Lemma 2.1. *Let $n \geq 2$, and let Y be a multi-set of n integers from $[0, n]$ with $1 \in Y$. Then there exists a partition of Y into two non-empty multi-sets L and R such that $\ell \geq |L|$ for any $\ell \in L$, and $r \leq |L|$ for any $r \in R$.*

Proof. Let $y_1 \geq y_2 \geq \cdots \geq y_n$ be an enumeration of the elements in Y. Since $1 \in Y$, we have $y_n = 1$ or $y_n = 0$ and therefore $y_n < n$. Moreover, $1 \in Y$ implies $y_1 \geq 1$. Let k be the smallest index with $y_k < k$. By the preceding observations $2 \leq k \leq n$. Then $y_{k-1} \geq k-1$ and $y_k \leq k-1$, and the multi-sets $L = \{\, y_i : i \leq k-1 \,\}$ and $R = \{\, y_i : i \geq k \,\}$ give the desired partition of Y, with $|L| = k-1$. □

Lemma 2.2. *For $n \geq 2$ players, the above protocol is $1/(2n-2)$-fair. For a single player, the above protocol is 1-fair.*

Proof. The statement will be proved by induction on the number n of players. The statement for $n = 1$ is trivial.

We first describe the winning strategies for the players p with $2 \leq p \leq n$. The only decision made by this player is the choice of the integer x_p in step (S2). We claim that

$$x_p = \begin{cases} 0 & \text{if } (2n-2)\mu_p(C_L) < \mu_p(C) \\ n & \text{if } (2n-2)\mu_p(C_L) > (2n-3)\mu_p(C) \\ \lceil (n-1) \cdot \mu_p(C_L)/\mu_p(C) \rceil & \text{otherwise} \end{cases}$$

is a good choice for him. Consider the case where in step (S3) the protocol assigns player p to the group L. Since $x_p \geq |L| \geq 1$ holds, $x_p = 0$ is not possible in this case. This yields $\mu_p(C) \leq (2n-2)\mu_p(C_L)$. If $|L| = 1$, then p will receive the whole piece C_L that has measure at least $\mu_p(C)/(2n-2)$. If $|L| \geq 2$ and $x_p < n$,, then player p by induction may enforce to receive a piece of measure at least

$$\frac{\mu_p(C_L)}{2|L|-2} \geq \frac{\mu_p(C_L)}{2x_p-2} \geq \frac{\mu_p(C_L)}{2(n-1)\mu_p(C_L)/\mu_p(C)} = \frac{\mu_p(C)}{2n-2}.$$

Finally, if $|L| \geq 2$ and $x_p = n$, then player p by induction may enforce to receive at least

$$\frac{\mu_p(C_L)}{2|L|-2} \geq \frac{\mu_p(C_L)}{2n-4} \geq \frac{\mu_p(C)}{2n-2}.$$

Next consider the case where the protocol assigns player p to R. Since $1 \leq |R| = n - |L| \leq n - x_p$, $x_p = n$ is not possible in this case. This yields $(2n-2)\mu_p(C_L) \leq (2n-3)\mu_p(C)$. If $|R| = 1$, then p will receive the whole piece C_R that has measure

$$\mu_p(C_R) = \mu_p(C) - \mu_p(C_L) \geq \frac{\mu_p(C)}{2n-2}.$$

If $|R| \geq 2$, and $x_p > 0$, then player p by induction can force to get a piece of measure at least

$$\frac{\mu_p(C_R)}{2|R|-2} \geq \frac{\mu_p(C_R)}{2n-2x_p-2} \geq \frac{\mu_p(C) - \mu_p(C_L)}{2n-2(n-1)\mu_p(C_l)/\mu_p(C)-2} = \frac{\mu_p(C)}{2n-2}.$$

Finally if $|R| \geq 2$ and $x_p = 0$, then player p by induction can enforce to get at least

$$\frac{\mu_p(C_R)}{2|R|-2} \geq \frac{\mu_p(C_R)}{2n-4} \geq \frac{\mu_p(C)}{2n-2}.$$

To summarize, in all possible cases player p can get at least a fraction $1/(2n-2)$ of $\mu_p(C)$.

Finally, we describe a winning strategy for the first player. The only decision made by this player is how to cut the cake in Step (S1). We claim that cutting the cake such that $\mu_1(C_L) = \mu_1(C)/(2n-2)$ is a good choice for him. If the first player is assigned to group L, he is the only player in L and gets the whole piece C_L of measure at least $\mu_1(C)/(2n-2)$. If the first player is assigned to group R, he shares C_R with at most $n-2$ other players. By induction, he can enforce to get a piece of measure at least

$$\frac{\mu_1(C_R)}{2n-4} = \frac{(2n-3)\mu_1(C)/(2n-2)}{2n-3} > \frac{\mu_1(C)}{2n-2}.$$

This completes the proof of Lemma 2.2, and it also completes the proof of the positive statement in Theorem 1.1. □

3. Proof of the lower bound

In this section, we prove the negative statement in Theorem 1.1. Suppose for the sake of contradiction that for some real number $\beta > 1/(2n-2)$, there exists a β-fair protocol for n players that uses only $n-1$ cuts. We investigate the situation where all players behave according to their winning β-strategies. Without loss of generality we normalize the measures such that $\mu_p(C) = 1$ holds for all players p. Whenever a piece X is cut into two pieces X_L and X_R, we fix the measures $\mu_p(X_L)$ and $\mu_p(X_R)$ for all players as follows:

- For the cutter p, his winning strategy fully determines $\mu_p(X_L)$ and $\mu_p(X_R)$.
- If $\mu_p(X_L) \geq \mu_p(X_R)$, then for every player $q \neq p$ we fix

  $$\mu_q(X_L) = \min\{1/(n-1), \mu_q(X)\}$$

  and

  $$\mu_q(X_R) = \mu_q(X) - \mu_q(X_L).$$

  If $\mu_p(X_L) \leq \mu_p(X_R)$, then for $q \neq p$ we fix $\mu_q(X_R) = \min\{1/(n-1), \mu_q(X)\}$ and $\mu_q(X_L) = \mu_q(X) - \mu_q(X_R)$.

A piece X of the cake is *dangerous*, if $\mu_p(X) \leq 1/(n-1)$ holds for all players p except at most one. A piece X is *safe* if it is not dangerous. By the above fixing of the measures $\mu_q(X_L)$ and $\mu_q(X_R)$, at least one of the pieces X_L and X_R is dangerous.

Lemma 3.1. *Assume that at some moment in time, the protocol orders player p to cut some piece X of the cake. If the protocol is β-fair, then X must be a safe piece.*

Proof. Suppose that X is a dangerous piece, and let X_L and X_R be the two new pieces that result from p's β-strategy. First consider the case where $\mu_p(X) \leq 1/(n-1)$. Assume without loss of generality that $\mu_p(X_L) \geq \mu_p(X_R)$ and hence $\mu_p(X_R) \leq 1/(2n-2)$. Then for every player $q \neq p$, the cut of player p may yield $\mu_q(X_R) = 0$ and $\mu_q(X_L) = \mu_q(X)$. But then the measure of X_R is less or equal to $1/(2n-2)$ for all players. As X_R is unacceptable to all players, that is a contradiction.

Next consider the case where $\mu_p(X) \geq 1/(n-1)$. Since X is a dangerous piece, $\mu_q(X) \leq 1/(n-1)$ holds for every player $q \neq p$. In this case the cut of player p may yield $\mu_q(X_L) \leq 1/(2n-2)$ and $\mu_q(X_R) \leq 1/(2n-2)$ for every player $q \neq p$. Since at least one of X_L and X_R (or a subpiece of them) must go to a player $q \neq p$, this player q cannot avoid getting a piece of measure at most $1/(2n-2)$ although he was following his β-strategy. That is again a contradiction. □

Lemma 3.2. *If the measures are fixed as described above, then at any moment in time there is at most one safe piece.*

Proof. In the beginning, the cake C is the only safe piece. By Lemma 3.1, every cut must subdivide a safe piece. By our fixing of the measures, at least one of the two resulting pieces will be dangerous and at most one will be safe. □

Lemma 3.3. *If after k uts with $0 \leq k \leq n-2$ there still exists a safe piece X, then $\mu_p(X) \leq (n-k-1)/(n-1)$ holds for all players p.*

Proof. The proof is by induction on k. For $k = 0$, there is nothing to show since $\mu_p(C) = 1$ for all players p. So assume that the statement holds after $k-1$ cuts, and that player p does the kth cut and thereby produces the pieces X_L and X_R with $\mu_p(X_L) \geq \mu_p(X_R)$. Then X_L will be dangerous, and only X_R might be safe. By the inductive hypothesis, for the cutter p we have

$$\mu_p(X_R) \leq \frac{1}{2} \cdot \mu_p(X) \leq \frac{1}{2} \cdot \frac{n-k-2}{n-1} \leq \frac{n-k-1}{n-1}.$$

By the fixing of the measures, for every player $q \neq p$ we have

$$\mu_q(X_R) = \mu_q(X) - \min\{1/(n-1), \mu_q(X)\} = \max\{0, \mu_q(X) - 1/(n-1)\}.$$

The case $\mu_q(X_R) = 0$ is fine. And if $\mu_q(X_R) = \mu_q(X) - 1/(n-1)$, then the inductive hypothesis implies $\mu_q(X) \leq (n-k-1)/(n-1)$. □

Now we are ready for the final contradiction: After $n-2$ cuts have been made, there are $n-1$ pieces of cake. By Lemma 3.2 at most one of these pieces is safe. Suppose that X is a safe piece. Then by Lemma 3.3 it must satisfy $\mu_p(X) \leq 1/(n-1)$ for all players p; but this exactly means that X is dangerous. Consequently all $n-1$ pieces are dangerous, and by Lemma 3.1 the protocol has no possibility for making the final cut. This contradiction completes the proof of the negative statement in Theorem 1.1.

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