Hamburger Beiträge
zur Angewandten Mathematik

Continuous Convergence of Relations
- A Principle in Discretization Procedures -

Rainer Ansorge

Reihe A
Preprint 189
October 2005

Hamburger Beiträge zur Angewandten Mathematik

Reihe A	Preprints
Reihe B	Berichte
Reihe C	Mathematische Modelle und Simulation
Reihe D	Elektrische Netzwerke und Bauelemente
Reihe E	Scientific Computing
Reihe F	Computational Fluid Dynamics and Data Analysis

Continuous Convergence of Relations
– A Principle in Discretization Procedures –

R. Ansorge

Key words continuous convergence, relations, numerical procedures
MSC (2000) 65J05, 65M12

There are situations in Numerical Analysis concerning the convergence of discretization procedures which seem to be of different type but can uniformly be described by a property introduced here, called *continuous convergence of relations* . Historical examples demonstrate this standardization.

1 Introduction

As far as numerical procedures occur in Applied Analysis, i.e. replacement of a problem by a sequence of approximate problems, a fundamental property expected to be an inherent part of the methods is the convergence of the solutions of the approximate problems to a solution of the original problem under consideration.

This includes Approximation Theory, methods for the computation of approximate solutions of linear or nonlinear systems of ordinary or partial differential equations completed by initial and/or boundary conditions and by entropy conditions in the case of lost uniqueness, quadrature formulae etc.

Relations expected or guaranteed to be fulfilled by the solutions of the original problem have to be fulfilled in an at least similar manner by the approximate solutions. This leads to the necessity that the operators describing the original equations as well as additional relations known from the very beginning have to be approximated and have to converge in a suitable way to the original relations.

In this context, an abstract concept of so-called *continuous convergence* does occur in different situations of Numerical Analysis. This will here be demonstrated by means of historical examples.

2 Continuous Convergence

Roughly speaking, continuous convergnce of objects P_n acting on elements u^n $(n = 1, 2, \cdots)$ to an object P acting on an element u means the implcation

$$u^n \to u \quad \wedge \quad P_n u^n \to v \quad \Longrightarrow \quad Pu = v \quad . \tag{1}$$

We write

$$P_n \xrightarrow{c} P \ .$$

Continuous Convergence was at the first time introduced by du Bois-Reymond [2] in 1886 in a paper on the integration of series. Courant [3] used this concept 1914 in the theory of conformal mappings and Rinow ([13], p. 64) 1961 in connection with the convergence of sequences $\{C_n\}$ of continuous operators −mapping a metric space into a metric space− to a continuous limit operator C. Rinow proved the following theorem:

Theorem: (Rinow): *Let $(\mathcal{V}, \rho)$ and $(\mathcal{W}, \sigma)$ be metric spaces and $C : \mathcal{V} \to \mathcal{W}$ a continuous operator. Let $\{C_n | C_n : \mathcal{V} \to \mathcal{W}\}$ be a sequence of continuous operators. Then the following two properties are necessary and sufficient for $C_n \xrightarrow{c} C$:*

- a) *the operators C_n are equicontinuous*
- b) *$\{C_n\}$ converges pointwise to C on a subset $\tilde{\mathcal{V}}$ which is dense in $\mathcal{V}$.*

If $\mathcal{V}$ and $\mathcal{W}$ are Banach spaces and if the operators C and $C_n \quad (n = 1, 2, \cdots)$ are linear, equicontinuity of the operators C_n and the property of the operators to be uniformly bounded, coincide, and Rinow's theorem coincides with the Banach-Steinhaus theorem [1] so that in this case already pointwise convergence of $\{C_n\}$ to C on $\mathcal{V}$ implies continuous convergence.

Completeness or linearity of the spaces are not required for the validity of Rinow's theorem !

A particular application of the Banach-Steinhaus theorem is the Lax-Richtmyer theorem [9] on the convergence of consistent finite difference methods for linear partial evolution equations [1].

Another application of the Banach-Steinhaus theorem is the convergence (hence, even continuous convergence) of Gauss-type quadrature formulas.

Let us now generally assume that there is a metric space $\mathcal{V}$, an index set $\mathcal{J}$ and a relation R so that for ordered pairs $(u, \Phi) \in \mathcal{V} \times \mathcal{J}$ the question can be answered whether (u, Φ) belongs to $R \subset \mathcal{V} \times \mathcal{J}$ or not. If the relation is fulfilled, i.e. if $(u, \Phi) \in R$, we write

$$uR\Phi\,. \tag{2}$$

Moreover, assume that there are subsets $\mathcal{V}_n \subset \mathcal{V}$ $(n = 1, 2, \cdots)$ and relations R_n concerning ordered pairs $(u^n, \Phi) \in \mathcal{V}_n \times \mathcal{J}$ $(n = 1, 2, \cdots)$ so that the question arises whether or not u^n is related to Φ. If the answer is *yes*, i.e. if $(u^n, \Phi) \in R_n$, we write

$$u^n R_n \Phi\,. \tag{3}$$

Definition: We call the sequence $\{R_n\}$ of relations *continuously convergent* to the relation R with respect to the triple $(\mathcal{V}, \{\mathcal{V}_n\}, \mathcal{J})$ if the following implication holds:

$$\forall \Phi \in \mathcal{J} :\ \{u^n | u^n \in \mathcal{V}_n\ (n = 1, 2, \cdots)\,;\ u^n R_n \Phi\} \to u \quad \Longrightarrow \quad uR\Phi\,.$$

We then write

$$R_n \xrightarrow{c} R\,. \tag{4}$$

Trivial example:

Let $\mathcal{V} = \mathcal{R}$ (real numbers) and $\mathcal{J} = \{\Phi\}$ with a particular $\Phi \in \mathcal{R}$.

Define: $\quad uR\Phi \Longleftrightarrow u \leq \Phi$.

[1] see subsection 3.2 of this paper

Let $\mathcal{V}_n \subset \mathcal{R} \quad (n = 1, 2, \cdots)$ and define R_n by

$$u^n R_n \Phi \quad \Longleftrightarrow \quad u^n \leq \Phi + \frac{1}{n} \ .$$

Then, obviously,

$$\{u^n \,|\, u^n R_n \Phi\} \to u \quad \text{leads to} \quad u \leq \Phi \,,$$

i.e. to

$$uR\Phi \ .$$

3 Realizations

If a problem to be treated consists in finding a solution u that fulfills uRf for given elements f of a certain set, if the approximate solutions u^n fulfill the relations $u^n R_n f \quad (n = 1, 2, \cdots)$, if −moreover− an existence theorem for the solution u is available as well as a convergence theorem $u^n \to u$ of the numerical method, $R_n \overset{c}{\to} R$ does only reflect this convergence theorem.

The next subsection shows an example:

3.1 Approximation Theory

Consider the situation $\mathcal{V} = C([0, 1])$ equipped with the Tschebychef norm $\|u\|_\infty = \max\limits_{0 \leq t \leq 1} |u(t)|$.

Let $\mathcal{J} = \mathcal{V}$ and $\mathcal{V}_n = \mathcal{P}_n$ (polynomials of maximal degree n).

Define R by

$$uRf \quad \Longleftrightarrow \quad u \equiv f$$

and R_n by

$$u^n R_n f \quad \Longleftrightarrow \quad u^n(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} f\left(\frac{\nu}{n}\right) x^\nu (1-x)^{n-\nu} \quad (n = 1, 2, \cdots) \ . \tag{5}$$

Here, the right hand side of (5) is monotone with respect to f for each fixed $n \in \mathcal{N}$, and because of

$$u^n(x) = \begin{cases} 1 & \text{for} \quad f(t) \equiv 1 \\ x & \text{for} \quad f(t) = t \\ x^2 + x\frac{1-x}{n} & \text{for} \quad f(t) = t^2 \end{cases} \ , \tag{6}$$

$$\lim_{n \to \infty} \|u^n - f\|_\infty = 0 \tag{7}$$

holds in all three cases.

But from Korovkin's theorem [7], (6) leads to the fact that (7) does not only hold in these special cases but **for all** $f \in \mathcal{J}$.

Hence,

$$\{u^n | u^n R_n f\} \to u$$

yields $u \equiv f$, i.e. uRf .