DESY-06-032
SFB/CPP-06-13

Hypergeometric representation of the two-loop equal mass sunrise diagram

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Abstract

A recurrence relation between equal mass two-loop sunrise diagrams differing in dimensionality by 2 is derived and it's solution in terms of Gauss' ${}_2F_1$ and Appell's F_2 hypergeometric functions is presented. For arbitrary space-time dimension d the imaginary part of the diagram on the cut is found to be the ${}_2F_1$ hypergeometric function with argument proportional to the maximum of the Kibble cubic form. The analytic expression for the threshold value of the diagram in terms of the hypergeometric function ${}_3F_2$ of argument $-1/3$ is given.

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1 Introduction

The evaluation of radiative corrections for modern high precision particle physics is becoming a more and more demanding task. Without inventing new mathematical methods and new computer algorithms the progress in calculating multi-loop, multi-leg Feynman diagrams depending on several momentum and mass scales will be not possible.

An important class of radiative corrections comes from self-energy type of Feynman diagrams, which also occur in evaluating vertex, box and higher multileg diagrams. At the two-loop level different approaches for calculating self-energy diagrams are available [1]. A general algorithm for reduction of propagator type of diagrams to a minimal set of master integrals was proposed in [2]. This recurrence relations algorithm has been implemented in computer packages in [3] and [4],[5]. At present the most advanced package available for calculating two-loop self-energy diagrams with arbitrary massive particles was written by Stephen Martin and David Robertson [5]. It includes procedures for numerical evaluation of master integrals with arbitrary masses and also a database of analytically known master integrals. Integral representation for master integrals with arbitrary masses in four dimensional space time was proposed in [6].

Despite intensive efforts by many authors not all two-loop self-energy integrals with a mass are known analytically. Even the imaginary part of the simplest sunrise self-energy diagram with three equal mass propagators was not known for arbitrary space-time dimension d until now. The two-loop sunrise integral with three equal masses was investigated in many publications [7]- [15]. Small and large momentum expansions of this integral for arbitrary space-time dimension d can be found in [11]. It's threshold expansion was given in [14]. A numerical procedure for evaluating the sunrise integral was described in [13]. The very latest effort of an analytic calculation of the diagram, by using the differential equation approach [16], was undertaken in Ref. [17].

It is the purpose of this paper to describe a new method and to present the analytic result for the equal mass two-loop sunrise master integral. To accomplish our goal we use the method of evaluation of master integrals by dimensional recurrences proposed in [18]. The application of this method to one-loop integrals was presented in [19] and [20]. In the present publication we extend the method to two-loop integrals.

As was already discovered in the one-loop case, the solutions of dimensional recurrences are combinations of hypergeometric functions. The knowledge of the hypergeometric representation of an integral means that we possess the most complete mathematical information available. This information can be effectively used in several respects. First, through analytic continuation formulae, the hypergeometric functions valid in one kinematic domain can be re-expressed in a different kinematic region. Second, these hypergeometric functions often have integral representation themselves, in which an expansion in $\varepsilon = (4-d)/2$ can be made, yielding expressions in logarithms, dilogarithms, elliptic integrals, etc.. Since very similar hypergeometric functions come from different kind of Feynman integrals the ε expansion derived in solving one problem can be used in other applications. Essential progress in the ε expansion of hypergeometric functions encountered in evaluating Feynman diagrams was achieved in [11],[13], [21] and [22]. Third, because the hypergeometric series is convergent and well behaved in a particular region of kinematical variables, it can be numerically evaluated [23], [24]. In addition a hypergeometric representation allows an asymptotic expansion of the integral in terms of ratios of different Gram determinants or ratios of momentum and mass scales which can provide fast numerical convergence of the result.

Our paper is organized as follows. In Sec. 2, we present the relevant difference equation connecting sunrise integrals with dimensionality differing by 2 as well as the differential equation for this integral. In Sec. 3. the method for finding the full solution of the dimensional recurrency is elaborated. Explicit expression for the sunrise integral in terms of Appell's function F_2 and Gauss' hypergeometric function ${}_2F_1$ is constructed in Sec. 4 and in Sec. 5 the differential equation approach and the method of dimensional recurrences is compared. In the Appendix some useful formulae for the hypergeometric functions ${}_2F_1$, F_1 and F_2 are given together with their integral representations.

2 Difference and differential equations for the sunrise integral

The generic two-loop self-energy type diagram in d dimensional Minkowski space with three equal mass propagators is given by the following integral:

$$J_3^{(d)}(\nu_1,\nu_2,\nu_3) \equiv \int\int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2}{(i\pi^{d/2})^2} \frac{1}{(k_1^2-m^2)^{\nu_1}((k_1-k_2)^2-m^2)^{\nu_2}((k_2-q)^2-m^2)^{\nu_3}}. \tag{2.1}$$

For integer values of ν_j the integrals (2.1) can be expressed in terms of only three basis integrals $J_3^{(d)}(1,1,1)$, $J_3^{(d)}(2,1,1)$ and $J_3^{(d)}(0,1,1) = (T_1^{(d)}(m^2))^2$ where

$$T_1^{(d)}(m^2) = \int \frac{d^d k}{[i\pi^{\frac{d}{2}}]} \frac{1}{k^2-m^2} \;=\; -\Gamma\left(1-\frac{d}{2}\right) m^{d-2}. \tag{2.2}$$

The relation connecting $d-2$ and d dimensional integrals $J_3^{(d)}(\nu_1,\nu_2,\nu_3)$ follows from the relationship given in Ref. [2]:

$$\begin{aligned} J_3^{(d-2)}(\nu_1,\nu_2,\nu_3) &= \nu_1\nu_2 J_3^{(d)}(\nu_1+1,\nu_2+1,\nu_3) \\ &+ \nu_1\nu_3 J_3^{(d)}(\nu_1+1,\nu_2,\nu_3+1) + \nu_2\nu_3 J_3^{(d)}(\nu_1,\nu_2+1,\nu_3+1). \end{aligned} \tag{2.3}$$

Relation (2.3) taken at $\nu_1=\nu_2=\nu_3=1$ and $\nu_1=2,\ \nu_2=\nu_3=1$ gives two equations. To simplify these equations we use the recurrence relations proposed in [2]. From these two equations by shifting $d\to d+2$ two more relations follow. They are used to exclude $J_3^{(d)}(2,1,1)$ from one of the relations, so that we obtain a difference equation for the master integral $J_3^{(d)}(1,1,1)\equiv J_3^{(d)}$:

$$\begin{aligned} &12z^3(d+1)(d-1)(3d+4)(3d+2) && J_3^{(d+4)} \\ &-4m^4(1-3z)(1-42z+9z^2)z(d-1)d && J_3^{(d+2)} \\ &-4m^8(1-z)^2(1-9z)^2 && J_3^{(d)} \\ &= 3z[(z+1)(27z^2+18z-1)d^2-4z(1+9z)d-48z^2]m^{2d+2} && \Gamma\left(-\frac{d}{2}\right)^2, \end{aligned} \tag{2.4}$$

where

$$z = \frac{m^2}{q^2}. \tag{2.5}$$

The integral $J_3^{(d)}$ satisfies also a second order differential equation [11]. Taking the second derivative of $J_3^{(d)}$ with respect to mass gives

$$\frac{d^2}{dm^2\ dm^2} J_3^{(d)}(1,1,1) = 6J_3^{(d)}(2,2,1) + 6J_3^{(d)}(3,1,1). \tag{2.6}$$

Again using the recurrence relations from [2], the integrals on the r.h.s can be reduced to the same three basis integrals. Using

$$J_3^{(d)}(2,1,1) = \frac{1}{3}\ \frac{d}{dm^2} J_3^{(d)}(1,1,1) \tag{2.7}$$

from (2.6) we obtain:

$$\begin{aligned} 2(1-z)(1-9z)z^2\frac{d^2 J_3^{(d)}}{dz^2} &- z[9z^2(d-4)+10z(d-2)+8-3d]\frac{dJ_3^{(d)}}{dz} \\ &+ (d-3)[z(d+4)+d-4]J_3^{(d)} = 12zm^{(2d-6)}\Gamma^2\left(2-\frac{d}{2}\right). \end{aligned} \tag{2.8}$$

The differential equation (2.8) will be used in Sec.4 to find the momentum dependence of arbitrary periodic constants in the solution of the difference equation (2.4).

3 Solution of the dimensional recurrency

Equation (2.4) is a second order inhomogeneous equation with polynomial coefficients in d. The full solution of this equation is given by (see Ref. [25] and references therein):

$$J_3^{(d)} = J_{3p}^{(d)} + \widetilde{w}_a(d) J_{3a}^{(d)} + \widetilde{w}_b(d) J_{3b}^{(d)}, \tag{3.9}$$

where $J_{3p}^{(d)}$ is a particular solution of (2.4), $J_{3a}^{(d)}, J_{3b}^{(d)}$ is a fundamental system of solutions of the associated homogeneous equation and $\widetilde{w}_a(d)$, $\widetilde{w}_b(d)$ are arbitrary periodic functions of d satisfying relations:

$$\widetilde{w}_a(d+2) = \widetilde{w}_a(d), \qquad \widetilde{w}_b(d+2) = \widetilde{w}_b(d). \tag{3.10}$$

The order of the polynomials in d of the associated homogeneous difference equation can be reduced by making the substitution

$$J_3^{(d)} = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{3d}{2}-3\right)\Gamma\left(\frac{d-1}{2}\right)} \overline{J}_3^{(d)}. \tag{3.11}$$

The associated homogeneous equation for $\overline{J}_3^{(d)}$ takes the simpler form

$$\frac{16z^3}{27m^8(1-z)^2(1-9z)^2}\,\overline{J}_3^{(d+4)} - \frac{2d(1-3z)(1-42z+9z^2)z}{27m^4(1-z)^2(1-9z)^2}\,\overline{J}_3^{(d+2)} - \frac{(3d-2)(3d-4)}{36}\,\overline{J}_3^{(d)} = 0. \tag{3.12}$$

Putting

$$d = 2k - 2\varepsilon, \qquad y^{(k)} = \rho^{-k}\,\overline{J}_3^{(2k-2\varepsilon)}, \tag{3.13}$$

we transform Eq.(3.12) to a standard form

$$A\rho^2 y^{(k+2)} + (B + C\ k)\rho y^{(k+1)} - (\alpha+k)(\beta+k) y^{(k)} = 0, \tag{3.14}$$

where

$$\begin{aligned} A &= \frac{16z^3}{27m^8(1-z)^2(1-9z)^2}, \qquad B = \frac{4\varepsilon}{27}\frac{(1-3z)(1-42z+9z^2)z}{m^4(1-z)^2(1-9z)^2}, \\ C &= -\frac{B}{\varepsilon}, \qquad \alpha = -\varepsilon - \frac{1}{3}, \qquad \beta = -\varepsilon - \frac{2}{3}, \end{aligned} \tag{3.15}$$

and ρ is for the time being, an arbitrary constant. In order to get Eq.(3.14) into a more convenient form, we will define three parameters ρ, x and γ by the equations

$$A\rho^2 = x(1-x), \quad B\rho = \gamma - (\alpha+\beta+1)x, \quad C\rho = 1-2x. \tag{3.16}$$

These have the solution

$$x = \frac{1-2C\rho}{2} = \frac{(1-9z)^2}{(1+3z)^3} = \frac{q^2(q^2-9m^2)^2}{(q^2+3m^2)^3}, \tag{3.17}$$

$$\rho = \frac{1}{\sqrt{4A+C^2}} = \frac{27}{4}\frac{m^4(1-z)^2(1-9z)^2}{z(1+3z)^3} = \frac{27}{4}\frac{m^2(q^2-m^2)^2(q^2-9m^2)^2}{(q^2+3m^2)^3}, \tag{3.18}$$

$$\gamma = B\rho + (\alpha+\beta+1)x = -\varepsilon, \tag{3.19}$$

and Eq. (3.14) can accordingly be written in the form

$$x(1-x)y^{(k+2)} + [(1-2x)k + \gamma - (\alpha+\beta+1)x]y^{(k+1)} - (\alpha+k)(\beta+k)y^{(k)} = 0. \tag{3.20}$$

The fundamental system of solutions of this equation consists of two hypergeometric functions [25]. For example, in the case when $|1-x| < 1$ (large q^2) the solutions are

$$
\begin{aligned}
y_1^{(k)} &= (-1)^k \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\alpha+\beta-\gamma+k+1)} \; {}_2F_1(\alpha+k, \beta+k, \alpha+\beta-\gamma+k+1; 1-x), \\
y_2^{(k)} &= \frac{\Gamma(\alpha+\beta-\gamma+k)}{(1-x)^k} \; {}_2F_1(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1-k; 1-x).
\end{aligned}
\tag{3.21}
$$

Once we know the solutions of the homogeneous equation, a particular solution $J_{3p}^{(d)}$ can be obtained by using Lagrange's method of variation of parameters. Lagrange method for finding a particular solution is well described in [25]. The application of the method is straightforward but tedious. Explicit result will be given in the next section.

It is interesting to note that the argument of the Gauss' hypergeometric function is related to the maximum of the Kibble cubic form [26]:

$$
\Phi(s,t,u) = stu - (s+t+u)m^2(m^2+q^2) + 2m^4(m^2+3q^2), \tag{3.22}
$$

provided that the following condition is satisfied:

$$
s+t+u = q^2 + 3m^2. \tag{3.23}
$$

The maximal value $\Phi_{\max} = \frac{1}{27}\, q^2(q^2-9m^2)^2$ occurs at $s=t=u=\frac{1}{3}\,(q^2+3m^2)$ and we see that the kinematical variable (3.17) can be written as

$$
x = \frac{\Phi(s,t,u)}{stu}\Big|_{s=t=u=\frac{1}{3}\,(q^2+3m^2)} . \tag{3.24}
$$

This observation may be useful in finding the characteristic variable in the general mass case [27]. Also one can try to apply the method described above to find the imaginary part of the sunrise integral in the general mass case in arbitrary space-time dimension.

4 Explicit analytic expression for $J_3^{(d)}$

To find the full solution of Eq. (2.4) we assume that q^2 is large. The region of large momentum squared corresponds to $x \sim 1$ and therefore as a fundamental system of solutions of the homogeneous equation we take $y_1^{(k)}$ and $y_2^{(k)}$. According to (3.11), (3.13) and (3.21) the solution of the associated homogeneous difference equation will be of the form

$$
\begin{aligned}
J_{3,h}^{(d)} &= w_1(z) \frac{\Gamma\left(\frac{d}{2}-\frac{1}{3}\right)\Gamma\left(\frac{d}{2}-\frac{2}{3}\right)\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{3d}{2}-3\right)\Gamma\left(\frac{d-1}{2}\right)} \rho^{\frac{d}{2}} e^{i\pi\frac{d}{2}} \; {}_2F_1\left[\begin{matrix} \frac{d}{2}-\frac{1}{3}, \frac{d}{2}-\frac{2}{3}; \\ \frac{d}{2}; \end{matrix} \; 1-x\right] \\
&+ w_2(z) \frac{\Gamma^2\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{3d}{2}-3\right)\Gamma\left(\frac{d-1}{2}\right)} \frac{\rho^{\frac{d}{2}}}{(1-x)^{\frac{d}{2}}} \; {}_2F_1\left[\begin{matrix} \frac{1}{3}, \frac{2}{3}; \\ 2-\frac{d}{2}; \end{matrix} \; 1-x\right].
\end{aligned}
\tag{4.25}
$$

The arbitrary periodic functions $w_1(z)$ and $w_2(z)$ can be determined either from the $d \to \infty$ asymptotics or using the differential equation (2.8). Substituting (4.25) into (2.8) we obtain two simple equations

$$
\begin{aligned}
& z(1-z)(1+3z)(1-9z)\frac{dw_1(z)}{dz} - 2(1+6z-39z^2)w_1(z) = 0, \\
& z(1+3z)(1-9z)\frac{dw_2(z)}{dz} + 3(1-z)w_2(z) = 0.
\end{aligned}
\tag{4.26}
$$