MS-TP-04-1
DESY 04-015
hep-lat/0402003

Chiral perturbation theory for partially quenched twisted mass lattice QCD

Gernot Münster, Christian Schmidt
Institut für Theoretische Physik, Universität Münster
Wilhelm-Klemm-Str. 9, D-48149 Münster, Germany
e-mail: munsteg@uni-muenster.de, sschmidt@uni-muenster.de

Enno Scholz
Deutsches Elektronen-Synchrotron DESY
Notkestr. 85, D-22603 Hamburg, Germany
e-mail: enno.e.scholz@desy.de

February 3, 2004
(revised: October 01, 2004)

Abstract

Partially quenched Quantum Chromodynamics with Wilson fermions on a lattice is considered in the framework of chiral perturbation theory in the mesonic sector. Two degenerate quark flavours are associated with a chirally twisted mass term. The pion masses and decay constants are calculated in next-to-leading order including terms linear in the lattice spacing a.

PACS numbers: 11.15.Ha, 12.38.Gc, 12.39.Fe
Keywords: lattice gauge theory, quantum chromodynamics, chiral perturbation theory

A notorious problem for numerical simulations of Quantum Chromodynamics with dynamical quarks is to reach the region of realistic small quark masses. To deal with this problem different methods are being used. On the theoretical side, chiral perturbation theory supplies us with formulae which allow to extrapolate pion masses and other observables from medium to small quark masses [1, 2, 3]. The low energy parameters of chiral perturbation theory, the *Gasser-Leutwyler coefficients*, can in turn be determined by numerical simulations of lattice QCD, see [4] for a review. The effects of the finite lattice spacing a are taken into account in chiral perturbation theory in form of an expansion in powers of a [5, 6, 7, 8, 9]. In [10, 11, 12, 13, 14, 15] numerical results from lattice QCD are compared with chiral perturbation theory in next-to-leading order.

On the algorithmic side partially quenched QCD is an approach to the regime of small quark masses. In a numerical simulation of partially quenched QCD the Monte Carlo updates are being made with sea-quarks, which have large enough masses m_S in order to allow a tolerable simulation speed. On the other hand, quark propagators and related observables are evaluated with smaller valence-quark masses m_V. Chiral perturbation theory has been adopted to the case of partially quenched QCD in [16, 17].

Recently it has been advocated to employ a chirally twisted quark mass matrix for Wilson fermions [18, 19] in order to improve the efficiency of QCD simulations [20, 21]. At present, this approach appears to be very attractive for reducing lattice effects in numerical simulations. First numerical results using this approach are very promising [22]. The corresponding modifications of chiral perturbation theory for lattice QCD have been worked out to order a in [23].

Combining these approaches, which is a logical next step, it appears attractive to simulate QCD with chirally twisted quark masses in a partially quenched manner. For the theoretical analysis of the data it is desirable to extend the results of [23] to the partially quenched case. This is the object of this letter.

We consider lattice QCD with $N_f = 2$ flavours of sea quarks and the same number of valence quarks. For simplicity we restrict ourselves to the case of degenerate quark masses ($m_u = m_d = m$). The quark mass matrix contains two chiral twist angles. Chiral perturbation theory is applied to the mesonic sector. The pion masses and decay constants are calculated in chiral perturbation theory in next-to-leading (one-loop) order, including lattice terms linear in the lattice spacing a.

Recent Monte Carlo simulations of the partially quenched model [14, 15] show that the one-loop contributions of chiral perturbation theory (for vanishing twist) can be identified in the numerical data and used to estimate Gasser-Leutwyler coefficients. In particular, it turned out that partially quenched chiral perturbation theory for lattice QCD is already applicable for quark masses below $\frac{1}{2}m_s$.

A theoretical description of partially quenched QCD can be obtained through the introduction of ghost quarks [24]. For each valence quark a corresponding bosonic ghost quark is added to the model. The functional integral over the ghost quark fields then cancels the fermion determinant of the valence quarks and only the sea quark determinant remains in the measure. In our case there are 2 flavours of valence quarks, sea quarks and ghost quarks, each. The quark mass matrix is

$$M = \mathrm{diag}(m_V, m_V, m_S, m_S, m_V, m_V). \tag{1}$$

A chirally twisted mass matrix, depending on two twist angles can be introduced in the form

$$M(\omega_V, \omega_S) = M \mathrm{e}^{\mathrm{i}\,\omega_V \tau_3^V \gamma_5}\ \mathrm{e}^{\mathrm{i}\,\omega_S \tau_3^S \gamma_5}\ \mathrm{e}^{\mathrm{i}\,\omega_V \tau_3^G \gamma_5}\,, \tag{2}$$

where

$$\tau_3^V = \begin{pmatrix} \tau_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tau_3^S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tau_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tau_3^G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix}. \tag{3}$$

In chiral perturbation theory the dynamics of the pseudo-Goldstone fields is described by a low energy effective Lagrangian. For partially quenched QCD the pseudo-Goldstone fields, which we shall generally call pions, are parameterized by a graded matrix of the form

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{4}$$

The 4×4 matrix A and the 2×2 matrix D contain commuting numbers, whereas the 4×2 matrix B and the 2×4 matrix C have anticommuting entries. U is an element of the supergroup SU(4|2) and can be represented as

$$U(x) = \exp\left(\frac{\mathrm{i}}{F_0}\Phi(x)\right). \tag{5}$$

The commuting elements of the graded matrix Φ represent the pseudo-Goldstone bosons made from a quark and an anti-quark with equal statistics, and the anticommuting elements of Φ represent pseudo-Goldstone fermions which are built from one fermionic quark and one bosonic quark. The supertrace of Φ has to vanish,

$$\mathrm{sTr}\,\Phi = 0\,, \tag{6}$$

where one defines

$$\mathrm{sTr}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{Tr} A - \mathrm{Tr} D\,. \tag{7}$$

This condition introduces technical complications in loop calculations. One way to deal with them [25] is to drop the condition $\mathrm{sTr}\,\Phi = 0$, adding a mass term proportional to $(\mathrm{sTr}\,\Phi)^2$ and sending its mass to infinity in the end. Another way, which we followed in our calculations, is to choose a basis of 35 super-traceless generators T_j of SU(4|2) and to expand

$$\Phi = \sum_j \pi_j T_j\,. \tag{8}$$

The chiral effective Lagrangian to leading order, including lattice artifacts up to first order in the lattice spacing a is given by [7]

$$\mathcal{L}_2 = \frac{F_0^2}{4}\,\mathrm{sTr}\left(\partial_\mu U^\dagger \partial^\mu U\right) - \frac{F_0^2}{4}\,\mathrm{sTr}\left(\chi U^\dagger + U \chi^\dagger\right) - \frac{F_0^2}{4}\,\mathrm{sTr}\left(\rho U^\dagger + U \rho^\dagger\right). \tag{9}$$

The mass term contains the matrix

$$\chi = 2B_0 M \tag{10}$$

and the lattice term is parameterized by

$$\rho = 2W_0 a\,\mathbf{1}\,, \tag{11}$$

with two constants B_0 and W_0.

Similar to the unquenched case, the chiral twist can be incorporated in chiral perturbation theory by transforming the lattice term into

$$\rho(\omega) \equiv \rho(\omega_V, \omega_S) = \rho\,\mathrm{e}^{\mathrm{i}\,\omega_V \tau_3^V}\,\mathrm{e}^{\mathrm{i}\,\omega_S \tau_3^S}\,\mathrm{e}^{\mathrm{i}\,\omega_V \tau_3^G}\,. \tag{12}$$

The effective Lagrangian in next-to-leading order (with respect to masses and momenta in the usual Weinberg power counting scheme) is then given by [7]

$$\begin{aligned}
\mathcal{L}_4 \;=\;& \frac{F_0^2}{4}\,\mathrm{sTr}\left(\partial_\mu U \partial_\mu U^\dagger\right) - \frac{F_0^2}{4}\,\mathrm{sTr}\left(\chi U^\dagger + U\chi^\dagger\right) - \frac{F_0^2}{4}\,\mathrm{sTr}\left(\rho(\omega) U^\dagger + U\rho(\omega)^\dagger\right) \\
& - L_1 \left[\mathrm{sTr}\left(\partial_\mu U \partial_\mu U^\dagger\right)\right]^2 - L_2\,\mathrm{sTr}\left(\partial_\mu U \partial_\nu U^\dagger\right)\mathrm{sTr}\left(\partial_\mu U \partial_\nu U^\dagger\right) \\
& - L_3\,\mathrm{sTr}\left(\left[\partial_\mu U \partial_\mu U^\dagger\right]^2\right) + L_4\,\mathrm{sTr}\left(\partial_\mu U \partial_\mu U^\dagger\right)\mathrm{sTr}\left(\chi U^\dagger + U\chi^\dagger\right) \\
& + W_4\,\mathrm{sTr}\left(\partial_\mu U \partial_\mu U^\dagger\right)\mathrm{sTr}\left(\rho(\omega) U^\dagger + U\rho(\omega)^\dagger\right) + L_5\,\mathrm{sTr}\left(\partial_\mu U \partial_\mu U^\dagger\left[\chi U^\dagger + U\chi^\dagger\right]\right) \\
& + W_5\,\mathrm{sTr}\left(\partial_\mu U \partial_\mu U^\dagger\left[\rho(\omega) U^\dagger + U\rho(\omega)^\dagger\right]\right) - L_6\left[\mathrm{sTr}\left(\chi U^\dagger + U\chi^\dagger\right)\right]^2 \\
& - W_6\,\mathrm{sTr}\left(\chi U^\dagger + U\chi^\dagger\right)\mathrm{sTr}\left(\rho(\omega) U^\dagger + U\rho(\omega)^\dagger\right) - L_7\left[\mathrm{sTr}\left(\chi U^\dagger - U\chi^\dagger\right)\right]^2 \\
& - W_7\,\mathrm{sTr}\left(\chi U^\dagger - U\chi^\dagger\right)\mathrm{sTr}\left(\rho(\omega) U^\dagger - U\rho(\omega)^\dagger\right) - L_8\,\mathrm{sTr}\left(\chi U^\dagger \chi U^\dagger + U\chi^\dagger U\chi^\dagger\right) \\
& - W_8\,\mathrm{sTr}\left(\chi U^\dagger \rho(\omega) U^\dagger + U\rho(\omega)^\dagger U\chi^\dagger\right) + \mathcal{O}(a^2).
\end{aligned} \tag{13}$$

In order to calculate the pion masses and other physical quantities, the Lagrangian has to be expanded in terms of the fields contained in Φ. In the untwisted case the expansion is around the minimum (more precisely, the saddle point) at vanishing Φ. The chiral twist, however, introduces a shift of the expansion point, as was already observed in the unquenched case [23]. We have calculated this shift in next-to-leading order including terms linear in the lattice spacing a, and expanded the effective action around the shifted minimum.

From the resulting expression the propagators and pion decay constants can be calculated in a one-loop calculation. The results are known for the case of vanishing twist [7]. Therefore one has to account for the effects of the twist angles now. The lattice term $\mathrm{sTr}(\rho(\omega)U^\dagger + U\rho(\omega)^\dagger)$ yields the quadratic contribution $\mathrm{sTr}((\rho(\omega) + \rho(\omega)^\dagger)\Phi^2)$, which contains the combination

$$\rho(\omega) + \rho(\omega)^\dagger = 4W_0 a(\cos\omega_V \mathbf{1}^V + \cos\omega_S \mathbf{1}^S + \cos\omega_V \mathbf{1}^G)\,. \tag{14}$$

We see that in this term the twist amounts to a multiplication of the untwisted lattice terms by factors $\cos\omega_V$ and $\cos\omega_S$, respectively. We have checked all terms proportional to the coefficients W_i and found that for them the same rule holds.

In the one-loop calculation the propagators and vertices from the leading order effective Lagrangian enter. We considered these terms and found that for the propagator and 4-point vertices the twist produces $\cos\omega$-factors as above. The 3-point vertices are modified by $\sin\omega$-factors. The relevant Feynman diagrams, however, contain two of these vertices and thus contribute to order a^2 only.

To summarize, at order a the chiral twist amounts to adding factors $\cos\omega_V$ or $\cos\omega_S$ to the untwisted quantities. This rule will not apply to order a^2, where new vertices appear. Results for masses and decay constants for partially quenched QCD with $N_f = 3$ flavours have been given by Rupak and Shoresh [7]. The formulae for a general number N_S of sea quarks and $N_V = 2$ valence quarks are presented in [13]. They give masses and decay constants for flavour-charged pions. From these we obtain the results for the twisted case under consideration using the procedure explained above. For the the pion masses we find

$$\begin{aligned} m_{SS}^2 &= \chi_S + \rho\cos\omega_S \\ &+ \frac{1}{32\pi^2 F_0^2}(\chi_S + \rho\cos\omega_S)^2 \log\left[\frac{\chi_S + \rho\cos\omega_S}{\Lambda^2}\right] \\ &+ \frac{16}{F_0^2}(2L_6 - L_4)\chi_S^2 + \frac{8}{F_0^2}(2L_8 - L_5)\chi_S^2 + \frac{16}{F_0^2}(W_6 - L_4)\chi_S\rho\cos\omega_S \\ &+ \frac{16}{F_0^2}(W_6 - W_4)\chi_S\rho\cos\omega_S + \frac{8}{F_0^2}(2W_8 - W_5 - L_5)\chi_S\rho\cos\omega_S \end{aligned} \tag{15}$$

$$\begin{aligned} m_{VV}^2 &= \chi_V + \rho\cos\omega_V \\ &+ \frac{1}{32\pi^2 F_0^2}(\chi_V + \rho\cos\omega_V)\Big\{\chi_V - \chi_S + \rho\cos\omega_V - \rho\cos\omega_S \\ &+ \Big(2\chi_V - \chi_S + 2\rho\cos\omega_V - \rho\cos\omega_S\Big)\log\left[\frac{\chi_V + \rho\cos\omega_V}{\Lambda^2}\right]\Big\} \\ &+ \frac{16}{F_0^2}(2L_6 - L_4)\chi_S\chi_V + \frac{8}{F_0^2}(2L_8 - L_5)\chi_V^2 + \frac{16}{F_0^2}(W_6 - L_4)\chi_S\rho\cos\omega_V \\ &+ \frac{16}{F_0^2}(W_6 - W_4)\chi_V\rho\cos\omega_S + \frac{8}{F_0^2}(2W_8 - W_5 - L_5)\chi_V\rho\cos\omega_V \end{aligned} \tag{16}$$

$$\begin{aligned} m_{VS}^2 &= \frac{1}{2}(\chi_V + \chi_S + \rho\cos\omega_V + \rho\cos\omega_S) \\ &+ \frac{1}{64\pi^2 F_0^2}(\chi_V + \chi_S + \rho\cos\omega_V + \rho\cos\omega_S)(\chi_V + \rho\cos\omega_V)\log\left[\frac{\chi_V + \rho\cos\omega_V}{\Lambda^2}\right] \\ &+ \frac{8}{F_0^2}(2L_6 - L_4)\chi_S(\chi_V + \chi_S) + \frac{2}{F_0^2}(2L_8 - L_5)(\chi_V + \chi_S)^2 \\ &+ \frac{8}{F_0^2}(W_6 - L_4)\chi_S(\rho\cos\omega_V + \rho\cos\omega_S) + \frac{8}{F_0^2}(W_6 - W_4)(\chi_V + \chi_S)\rho\cos\omega_S \\ &+ \frac{2}{F_0^2}(2W_8 - W_5 - L_5)(\chi_V + \chi_S)(\rho\cos\omega_V + \rho\cos\omega_S) \end{aligned} \tag{17}$$

and for the decay constants

$$\frac{F_{SS}}{F_0} = 1 - \frac{1}{16\pi^2 F_0^2}(\chi_S + \rho\cos\omega_S)\log\left[\frac{\chi_S + \rho\cos\omega_S}{\Lambda^2}\right]$$