Hamburger Beiträge
zur Angewandten Mathematik

Variational Discretization of Lavrentiev-Regularized State Constrained Elliptic Control Problems

M. Hinze, C. Meyer

Nr. 2007-18
November 2007

VARIATIONAL DISCRETIZATION OF LAVRENTIEV-REGULARIZED STATE CONSTRAINED ELLIPTIC CONTROL PROBLEMS

M. HINZE*, C. MEYER†

Abstract. In the present work, we apply a variational discretization proposed by the first author in [14] to Lavrentiev-regularized state constrained elliptic control problems. We extend the results of [18] and prove weak convergence of the adjoint states and multipliers of the regularized problems to their counterparts of the original problem. Further, we prove error estimates for finite element discretizations of the regularized problem and investigate the overall error imposed by the finite element discretization of the regularized problem compared to the continuous solution of the original problem. Finally we present numerical results which confirm our analytical findings.

Key words. Optimal control of elliptic equations, quadratic programming, pointwise state constraints, mixed constraints, Lavrentiev regularization

AMS subject classifications. 49K20, 49N10, 49M20

1. Introduction. In the present work, we apply variational discretization proposed by the first author in [14] to Lavrentiev-regularized state-constrained elliptic control problems. Let $\Omega \subset \mathbb{R}^n (n = 2, 3)$ denote an open, bounded domain with $C^{0,1}$-boundary Γ. As model problem, we consider for states $y \in Y := H^1(\Omega) \cap C(\bar{\Omega})$ and controls $u \in L^2(\Omega)$

$$
(\mathrm{P}) \quad \begin{cases} \text{minimize} & J(y,u) := \dfrac{1}{2}\displaystyle\int_\Omega |y - y_d|^2\,dx + \dfrac{\alpha}{2}\int_\Omega u^2\,dx \\ \text{subject to} & y = S\,u \text{ and } y(x) \le y_c(x) \text{ a.e. in } \Omega, \end{cases}
$$

where $y_d \in L^2(\Omega)$, $y_c \in C(\bar{\Omega})$ denote given functions, and $S : L^2(\Omega) \to Y$ denotes the control-to-state mapping, i.e. the solution operator of the Neumann problem

$$
-\Delta\, y + y = u \text{ in } \Omega \text{ and } \partial_n y = 0 \text{ on } \Gamma.
$$

Associated to (P) is the Lavrentiev-regularized control problem

$$
(\mathrm{P}_\lambda) \quad \begin{cases} \text{minimize} & J(y,u) := \dfrac{1}{2}\displaystyle\int_\Omega |y - y_d|^2\,dx + \dfrac{\alpha}{2}\int_\Omega u^2\,dx \\ \text{subject to} & y = S\,u \text{ and } \lambda u(x) + y(x) \le y_c(x) \text{ a.e. in } \Omega, \end{cases}
$$

where $\lambda > 0$ denotes the regularization parameter. Since the constraints in (P) and (P_λ), respectively, define closed convex sets, both problems admit unique solutions (y^*, u^*) and $(\bar{y}_\lambda, \bar{u}_\lambda)$.

The numerical treatment of problem (P) causes difficulties through the presence of the pointwise state constraints, since the corresponding Lagrange multiplier in general

*Mathematical Department, University Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany.

†Weierstrass Institute for Applied Analysis and Stochastics Mohrenstr. 39, D-10117 Berlin, Germany.

only represents a regular Borel measure (see Casas [7] or Alibert and Raymond [1]). In [18], Rösch, Tröltzsch, and the second author propose to circumvent these difficulties through approximating problem (P) by the family of problems (P_λ) ($\lambda > 0$). Among other things, they prove convergence of $(\bar{y}_\lambda, \bar{u}_\lambda) \to (y^*, u^*)$ in $L^2(\Omega)$ for $\lambda \to 0$. Furthermore, they show that the Lagrange multiplier associated to the mixed control–state constraint in (P_λ) is an L^2-function for every $\lambda > 0$. The development of numerical approaches to tackle problem (P) is ongoing [3, 17, 19]. An excellent overview can be found in [12, 13], where also further references are given.

Numerical analysis for problem (P) is presented by the first author and Deckelnick in [9]. Among other things, they prove convergence of finite element approximations to the control and to the state of order $1 - \varepsilon$ in two-dimensions, and of order $1/2 - \varepsilon$ in three dimensions, in L^2 and H^1, respectively. In [16], the second author obtains the same convergence order for piecewise constant approximations of the controls, and also extends these results to problems with additional box constraints on the control, compare also [11]. A general framework for numerical analysis of problems with pointwise state together with general constraints on the control is presented by Deckelnick and the first author in [10].

In the present paper, we extend the results of [18] for problem (P_λ) and prove weak convergence of the adjoint states p_λ in L^2 for λ tending to zero. Moreover, weak-$*$ convergence of the multipliers μ_λ in $C(\bar{\Omega})^*$ to their counterparts of problem (P) for $\lambda \downarrow 0$ is shown. Based on these results, we prove error estimates for variational discrete approximations to problem (P_λ). More precisely, in Theorem 3.8, we show

$$\|\bar{u}_\lambda - \bar{u}_{\lambda,h}\| + \|\bar{y}_\lambda - \bar{y}_{\lambda,h}\|_{H^1} \leq C h^{1-\frac{n}{4}}, \tag{1.1}$$

and

$$\|\bar{u}_\lambda - \bar{u}_{\lambda,h}\| + \|\bar{y}_\lambda - \bar{y}_{\lambda,h}\|_{H^1} \leq C\, \frac{1}{\lambda^2}\left(h^2 + \frac{1}{\lambda}\, h^3 + \frac{1}{\lambda^2}\, h^4\right) \tag{1.2}$$

is proven in Theorem 3.5. Here, $n = 2, 3$ denotes the space dimension and C is a generic positive constant independent of the finite element grid size h and of λ. To prove the first estimate we adapt the techniques developed in [10] for the analysis of the limit problem (P). The key idea of the proof of the second estimate consists in the fact that the substitution

$$v(x) = \lambda\, u(x) + y(x) \tag{1.3}$$

transforms (P_λ) into the purely control constrained optimal control problem

$$(\mathrm{PV}) \quad \begin{cases} \text{minimize} & \tilde{J}(y, v) := \dfrac{1}{2}\, \|y - y_d\|^2 + \dfrac{\alpha}{2\,\lambda^2}\, \|v - y\|^2 \\ \text{subject to} & -\Delta\, y + c_\lambda\, y = \dfrac{1}{\lambda}\, v \quad \text{in } \Omega \\ & \qquad \partial_n\, y = 0 \qquad \text{on } \Gamma \\ \text{and} & v(x) \leq y_c(x) \text{ a.e. in } \Omega. \end{cases}$$

Here, $c_\lambda := 1 + 1/\lambda$. Since (PV) is a purely control-constrained problem, it admits a unique Lagrange multiplier in $L^2(\Omega)$ associated to the inequality constraint. Moreover, the discretization techniques developed in [14] are directly applicable to

(PV) which is of major importance for the implementation of a semi–smooth Newton method for the numerical solution of (PV) and (P_λ), respectively. Furthermore, we also relate the finite element solution $(\bar{y}_{\lambda,h}, \bar{u}_{\lambda,h})$ to (y^*, u^*), i.e. the solution of the original purely state-constrained problem (P). Under the additional assumption that the solutions u_λ of (P_λ) are uniformly bounded in $L^\infty(\Omega)$, it follows by combining a result of [19] with (1.1) that

$$\|u^* - \bar{u}_{\lambda,h}\| \le C\Big(\sqrt{\lambda} + \max\{h\,|\log(h)|, h^{2-n/2}\}\Big), \tag{1.4}$$

while its combination with (1.2) implies

$$\|u^* - \bar{u}_{\lambda,h}\| \le C\,\Big(\sqrt{\lambda} + \frac{1}{\lambda^2}\big(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\big)\Big). \tag{1.5}$$

In view of (1.4) and (1.5), the overall error consists of two different contributions: one arising from the regularization and another one caused by the discretization. Moreover, from (1.5), we deduce that both error contributions behave contrarily with respect to λ (cf. Remark 3.7) which is also confirmed by our numerical findings (see Section 4). Hence, the optimal value of λ for a given mesh size h is larger than zero, and (1.4) indicates that the coupling $\lambda \sim h^2$ in case of $n = 2$ and $\lambda \sim h$ in three dimensions might be optimal (see Remark 3.10). Indeed, this result is also confirmed by our numerical observations.

The paper is organized as follows. In Section 2 we prove that, beside control and state, also the adjoint state and the Lagrange multipliers converge in some weaker sense to the solution of the original problem. Section 3 addresses the error analysis for the regularized problems and investigates how to couple λ and h. In Section 4, the numerical example is presented.

1.1. Notation. Throughout this article, we use the following notation. Given an open, bounded set $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, we denote by $(.\,,\,.)$ the natural inner product of in $L^2(\Omega)$. The corresponding norm is denoted by $\|.\|$. Moreover, for the dual pairing between $C(\bar{\Omega})$ and $C(\bar{\Omega})^*$, we write $\langle .\,,\,.\rangle$.

2. Weak convergence of the Lagrange multipliers. In the present section we prove convergence of the adjoint states and of the Lagrange multipliers of problem (P_λ) to their counterparts of problem (P). For this purpose it is convenient to introduce the reduced objective functional by $f(u) = J(S\,u, u)$ and the Lagrange functional $\mathcal{L} : L^2(\Omega) \times C(\bar{\Omega})^* \to \mathbb{R}$ by

$$\mathcal{L}(u, \mu) := f(u) + \langle S\,u - y_c\,,\,\mu\rangle.$$

Lagrange multipliers associated to the state constraint in (P) then are defined as follows:

Definition 2.1. *Let u^* denote the solution of* (P). *Then, $\mu \in C(\bar{\Omega})^*$ is called Lagrange multiplier, if it satisfies the following conditions:*

$$\frac{\partial \mathcal{L}}{\partial u}(u^*, \mu) = f'(u^*) + S^*\mu = 0 \tag{2.1}$$

$$\langle S\,u^* - y_c\,,\,\mu\rangle = 0 \tag{2.2}$$

$$\langle y\,,\,\mu\rangle \ge 0 \quad \forall\, y \in C(\bar{\Omega})^+, \tag{2.3}$$