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H. J. Oberle and R. Rosendahl

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H. J. Oberle • R. Rosendahl [1]

Abstract In this paper we consider general optimal control problems (OCP) which are characterized by a nonsmooth ordinary state differential equation. However, we allow only mild types of nonsmoothness. More precisely, we assume that the right-hand side of the state equation is piecewise smooth and that the switching points, which separate these pieces, are determined as points, where a state- and possibly control dependent (smooth) switching function changes sign. For this kind of optimal control problems necessary conditions are developed. Attention is payed to the situation that the switching function vanishes identically along a nontrivial subarc. Such subarcs, which we call singular state subarcs, are investigated with respect to necessary conditions and to junction conditions. In extension to earlier results, cf. [9], in this paper nonsmooth OCPs are considered with respect to the order of the switching function. Especially, the case of a zero-order switching function is included and examples of order zero, one and two are treated.

Key Words. Nonsmooth Optimal Control Problems, Necessary Conditions, Singular State Subarcs, Zermelo's Problem

1 Introduction

The paper is concerned with general optimal control problems (OCP) which are characterized by a nonsmooth ordinary state differential equation. More precisely, we assume that the right-hand side of the state equation is piecewise smooth and that the switching points, which separate these pieces, are determined as those points where a state- and possibly control-dependent (smooth) switching function changes sign. Nonsmooth optimal control problems of this type rarely have been mentioned in the literature, cf. for example [2, 6, 8]. Of course, they are special examples for the rather general theory of Clarke, [5]. Such problems sometimes occur in applications.

In a recent paper [9] the authors have considered an economic model due to Pohmer, cf. [11], for the optimal personal income distribution. The model is given in form of a nonsmooth OCP with two state variables (human capital, and capital) and three control variables which describe the consumption and the time allocation in time for working, education and recreation. In this model, the switching function turns out to be of order

[1] H.J. Oberle · R. Rosendahl
Department of Mathematics, University of Hamburg, Hamburg, Germany
e-mail: oberle@math.uni-hamburg.de · rosendahl@math.uni-hamburg.de

one. The OCP has been investigated with respect to necessary conditions. Especially the case of a so-called singular-state subarc has been considered.

In the present paper, we continue to consider nonsmooth OCPs. We include the case of an order-zero switching function and give necessary conditions for regular and singular OCPs of this type. Further, we consider two classical examples. The first example describes the optimal control of an electric circuit which includes a diode and a capacitor. This problem has already been investigated in the book of Clarke [5]. It is a nonsmooth OCP with a switching function of order zero. We apply our necessary conditions and present regular and singular solutions to this problem. By a slight modification - including a coil into the electric circuit - we obtain a nonsmooth OCP with an order-two switching function. For this problem we present regular solutions.

The second example is the classical Zermelo's navigation problem. Here, one has to determine optimal control functions for a time-minimal horizontal plane flight of an aircraft within a prescribed space-depending wind field. If we assume that the wind field contains certain lines of discontinuities (atmospheric fronts), we end up with a nonsmooth OCP with a switching function of order one. We apply the necessary conditions and present numerical solutions as well for the regular as for the singular case.

The paper is organized as follows: In the first part we consider a general nonsmooth OCP and derive corresponding necessary conditions in form of a multipoint boundary value problem. In section two, we further assume that the switching function along the solution trajectory changes sign only at isolated points (regularity assumption). The necessary conditions, we derive, differ for control dependent switching functions (order zero), on the one hand, and for switching functions which only depend on the state (positive order), on the other hand. In section three, in addition we admit singular state subarcs. Here, the necessary conditions can be derived only for order zero and order one problems. In the remaining three sections we investigate the examples, mentioned before.

2 Nonsmooth Optimal Control Problems, Regular Case

We consider a general OCP with a piecewise defined state differential equation. The problem has the following form.

Problem (P) Determine a piecewise continuous control function $u : [a, b] \to \mathbb{R}^m$, such that the functional

$$I = g(x(b)) \tag{1}$$

is minimized subject to the following constraints (state equations, boundary conditions, and control constraints)

$$x'(t) = f(x(t), u(t)), \quad t \in [a, b] \quad \text{a.e.}, \tag{2a}$$

$$r(x(a), x(b)) = 0, \tag{2b}$$

$$u(t) \in \mathcal{U} \subset \mathbb{R}^m. \tag{2c}$$

The control region $\mathcal{U}$ is assumed to be a compact and convex cuboid of the form $\mathcal{U} = \Pi_i[u_{i,\min}, u_{i,\max}]$. Further, we assume that the right-hand side of the state equation (2a) is of the special form

$$f(x,u) = \begin{cases} f_1(x,u), & \text{if} \quad S(x,u) < 0, \\ f_s(x,u), & \text{if} \quad S(x,u) = 0, \\ f_2(x,u), & \text{if} \quad S(x,u) > 0, \end{cases} \tag{3}$$

where the functions $S : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $f_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ ($k = 1, 2, s$), and $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^\ell$, $\ell \in \{0, \ldots, 2n\}$, are sufficiently smooth.

S is called the *switching function* of Problem (P). Note, that in many cases the dynamic f_s – the index s stands for *singular* – along the singular surface $S = 0$ will be given either by $f_s := f_1$ or by $f_s := f_2$.

Our aim is to derive necessary conditions for Problem (P). To this end, let (x^0, u^0) denote a solution of the problem with a piecewise continuous optimal control function u^0.

We assume that the problem is *regular* with respect to the minimum principle, that is: For suitable $\lambda,\ x \in \mathbb{R}^n$ the *Hamiltonians*

$$\mathcal{H}_j(x,u,\lambda) := \lambda^{\mathrm{T}} f_j(x,u), \quad j = 1, 2, s \tag{4}$$

possess a unique minimum u_j^0 with respect to the control $u \in \mathcal{U}$.

Finally, for this section, we assume that the following regularity assumption holds.

Regularity Condition (R) There exists a finite grid $a =: t_0 < t_1 < \ldots < t_q < t_{q+1} := b$ such that the optimal switching function $S[t] := S(x^0(t), u^0(t))$ is either positive or negative in each open subinterval $]t_{j-1}, t_j[, \quad j = 1, \ldots, q+1$.

Note, that the one-sided limits $u(t_j^\pm)$ exist due to the assumption of the piecewise continuity of the optimal control.

In the following, we distinguish two cases. If the switching function is independent on the control u, the switching function along the solution, $S[\cdot] := S(x^0(\cdot))$, is a continous function, so that t_j is a isolated root of $S[\cdot]$. We indicate this case by $p > 0$.

On the other hand, if the switching function depends explicitly on the control, $S[\cdot] := S(x^0(\cdot), u^0(\cdot))$ may have discontinuities at the t_j. In this case, we say that the switching function is of order zero, $p = 0$.

Now, we can summarize the necessary conditions for the OCP (P). Here, on each subinterval $[t_j, t_{j+1}]$, we denote $\mathcal{H}(x,u,\lambda) := \mathcal{H}_j(x,u,\lambda)$ where $j \in \{1, 2, s\}$ is chosen according to the sign of S in the corresponding (open) subinterval. The following theorem is a slight generalization of our previous results in the related paper [9].