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An Interior Point Method in Function Space for the Efficient Solution of State Constrained Optimal Control Problems [1]

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An Interior Point Method in Function Space for the Efficient Solution of State Constrained Optimal Control Problems †

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Abstract

We propose and analyse an interior point path-following method in function space for state constrained optimal control. Our emphasis is on proving convergence in function space and on constructing a practical path-following algorithm. In particular, the introduction of a pointwise damping step leads to a very efficient method, as verified by numerical experiments.
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1 Introduction

The construction and analysis of efficient algorithms for state constrained optimal control problems is still a considerable challenge. Presently, most popular methods that admit a (partial) analysis in function space are path-following methods, such as exterior penalty methods [7], Lavrentiev regularization [9, 10] and interior point methods [13, 14, 15]. Except for [13] and partially [10] (for a fixed Lavrentiev parameter) the available results are restricted to properties of the *homotopy path*, such as its existence, convergence and continuity. Except for these two works, not much is known about convergence of the associated *path-following algorithms*. This includes the important question if it is at all possible to follow the homotopy path by a practical algorithm, or if the sequence of iterates may stagnate far away from the desired solution. Closely connected and even more relevant from a practical point of view is the question how to choose homotopy parameters to obtain a fast and robust algorithm. These questions can certainly not be answered by an analysis of the path alone.

The aim of this paper is to propose and analyse an interior point method in function space that is capable of solving state constrained optimal control problems efficiently. The corresponding homotopy path has been analysed in [14, 15], so

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our emphasis here is on the Newton path-following method and on giving positive answers to the above questions. We establish qualitative convergence results in the following sense. Under suitable conditions there is a sequence of homotopy parameters μ_k that converges to 0 and a sequence of corresponding iterates x_k produced by a Newton corrector scheme that converges to the solution of the original problem x_*. The quantities used in the analysis, which yields convergence of the scheme as an a-priori result, can be modelled and estimated inside a numerical algorithm to yield a criterion for controlling the path-following algorithm efficiently. This is done in the spirit of [4, Chapter 5], but modified in a way that fits into our particular setting in function space.

To establish a rigorous analysis we essentially need estimates for two quantities. The first one, which reflects the most basic analytic properties of the homotopy path, is its local Lipschitz constant $\eta(\mu)$. The second captures the nonlinearity of the equations that define the homotopy path. This quantity, which governs the local convergence behaviour of Newton's method and in particular its radius of convergence, is an affine covariant Lipschitz constant for the Jacobian, denoted by $\omega(\mu)$. Since good a-posteriori estimates are available for η and ω, their role is not a purely analytic one, but they establish a close connection between a-priori theory and algorithmic implementation. In some sense, the algorithm is driven by an a-posteriori counterpart of the convergence theory established in this work.

Compared to [13] we introduce, as an algorithmic modification, a pointwise damping step, which prevents Newton's method from leaving the feasible domain and enhances the efficiency of the path-following scheme significantly. It is motivated by the idea to exploit the pointwise structure of the problem and has several useful interpretations. In our numerical experiments we observe that this modification allows the solution of state constrained optimal control problems in a few Newton steps.

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2 A Class of State Constrained Optimal Control Problems

Let Ω be an open and smoothly bounded domain in $\mathbb{R}^d$, $d = 1 \dots 3$ and $\overline{\Omega}$ its closure. Let Y denote the space of states and U the space of controls. Define $Z := Y \times U$ with $z := (y, u)$ and consider the following convex minimization problem, the details of which are fixed in the remaining section.

$$\min_{z \in Z} J(z) \quad \text{s.t. } Ay - Bu = 0 \qquad \underline{y} \leq y. \tag{1}$$

We set $Y = C(\overline{\Omega})$, and $U = L_2(Q)$ for a measurable set Q equipped with an appropriate norm. This setting includes optimal control problems subject to linear elliptic partial differential equations with distributed control ($Q = \Omega$), boundary control ($Q = \partial\Omega$) and finite dimensional control ($Q = \{1, \ldots, n\}$, equipped with the counting measure).

We will now specify our abstract theoretical framework, which holds throughout this work and collect a couple of basic results about this class of problems. Our framework is placed in the context of convex analysis, whose fundamentals can e.g. be looked up in [5].

Convex Functionals. For simplicity, let J be a quadratic tracking type functional with Tychonov regularization term:

$$J(z) = \frac{1}{2} \|y - y_d\|^2_{L_2(\Omega)} + \frac{\alpha}{2} \|u\|^2_{L_2(\Omega)}$$

Obviously, this functional is strictly convex and continuous in Z, and hence subdifferentiable. Its subdifferential is single valued and given by

$$Z^* \ni \partial J(z) = \begin{pmatrix} y - y_d \\ \alpha u \end{pmatrix}.$$

Equality Constraints. The equality constraint $Ay - Bu = 0$ is introduced to model a partial differential equation.

Let R be a reflexive Banach space and $B : U \to R$ be continuous. We assume that $A : Y \supset \operatorname{dom} A \to R$ is a linear operator, which is *densely defined, closed* that maps $\operatorname{dom} A$ to R *bijectively.*

In the context of optimal control R is often the dual of a Sobolev space and the operator B is usually defined as the adjoint of an embedding or a trace operator (cf. e.g. the discussion in [8] or [15]).

We consider A as a model of a differential operator, which may be unbounded. This depends of course on the choice of topology in Y. Closed, densely defined operators between Banach spaces are a classical concept of functional analysis. They generalize the concept of continuous operators and retain much of their structure. In particular, there is an open-mapping theorem, a closed range theorem, and adjoint operators are well defined. In this work and in [14, 15] only these basic properties of A are needed for a successful analysis. A classical introduction to unbounded operators is [6], but most elementary facts can also be found in standard textbooks on functional analysis.

There is a simple correspondence between a bijective closed operator and its inverse.

Lemma 2.1. *For Banach spaces Y and R let $A : Y \supset \operatorname{dom} A \to R$ be a linear operator. A is closed and bijective if and only if A possesses a continuous inverse $A^{-1} : R \to \operatorname{dom} A \subset Y$ in the sense that $A^{-1}A = id_{\operatorname{dom} A}$ and $AA^{-1} = id_R$.*