

Gauge Symmetry, T-Duality and Doubled Geometry

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Abstract

String compactifications with T-duality twists are revisited and the gauge algebra of the dimensionally reduced theories calculated. These reductions can be viewed as string theory on T-fold backgrounds, and can be formulated in a ‘doubled space’ in which each circle is supplemented by a T-dual circle to construct a geometry which is a doubled torus bundle over a circle. We discuss a conjectured extension to include T-duality on the base circle, and propose the introduction of a dual base coordinate, to give a doubled space which is locally the group manifold of the gauge group. Special cases include those in which the doubled group is a Drinfel'd double. This gives a framework to discuss backgrounds that are not even locally geometric.

1 Introduction

Two kinds of dimensional reduction of supergravities were proposed in the seminal paper of Scherk and Schwarz [1], each involving a twist by a group. Each gives a lower dimensional supergravity, which typically is gauged, i.e. has a non-Abelian Yang-Mills group. Recently it was understood how to lift these dimensional reductions to the full supergravity, string theory or M-theory [2, 3, 4, 5]. The key is to show that each can arise from a compactification, so that the full massive spectrum is defined, including Kaluza-Klein modes, massive string modes, wrapped branes etc. Understanding the compactification geometry is important in understanding the structure of the theory, and it turns out that this is intimately related to the structure of the gauge group. Some of the reductions, those with T-duality twists, do not lift to any compactification of supergravity. However, they can lift to non-geometric reductions of string theory on T-folds. Much remains to be understood about such non-geometric backgrounds, and the aim here is to use the gauge algebra of the dimensionally reduced theory to gain some insight into such reductions. In [6, 7], it was shown that string theory on a T-fold that looks like a T^d bundle locally has a natural formulation on a bundle in which the torus fibres are doubled to become T^{2d} . Our considerations here lead to a natural geometry in which all the dimensions are doubled, not just the fibres.

The first class of Scherk-Schwarz reductions look superficially like reductions on an n -torus, but twisted with the action of an n -dimensional group G . For this reason, they have become known, misleadingly, as twisted torus reductions. The reduction can be thought of as choosing an internal space that is the group manifold for G , which is typically non-compact, and then consistently truncating to fields independent of the ‘internal’ coordinates. In [3], it was shown that in most cases the same theory can be obtained from compactification on a compact manifold which looks like the group manifold locally. This requires the existence of a discrete subgroup $\Gamma \subset G$ such that $\mathcal{X} = G/\Gamma$ is compact (so that Γ is then a cocompact subgroup of G), in which case the theory is simply compactified on $\mathcal{X} = G/\Gamma$. The Scherk-Schwarz ansatz involves the expansion of the higher dimensional fields in terms of a basis of globally defined one-forms $\{\sigma\}$. In order for the one-forms σ to be globally defined on \mathcal{X} , it is necessary that they are invariant under the action of Γ , so that the reduction ansatz is invariant under Γ . However, one of the consequences of the reduction ansatz being invariant under Γ is that the gauged supergravity contains little information about the global structure of \mathcal{X} .

If we include a constant flux for the H -field so that $H \sim K_{mnp}\sigma^m \wedge \sigma^n \wedge \sigma^p + \dots$, the supergravity, resulting from such a Scherk-Schwarz compactification on \mathcal{X} , has gauge algebra

$$[Z_m, Z_n] = f_{mn}{}^p Z_p + K_{mnp} X^p \quad [X^m, Z_n] = f_{np}{}^m X^p \quad [X^m, X^n] = 0 \quad (1.1)$$

where Z_m generate isometries of \mathcal{X} and X^m generate antisymmetric tensor transformations for the B -field (see [3, 8] for details). Here $f_{mn}{}^p$ are the structure constants of the group G .

The other type of Scherk-Schwarz reduction starts with reduction on a torus T^d , so that the dimensionally reduced and truncated theory has a continuous duality symmetry K . This is followed by a further reduction on a circle with a duality twist, so that on going round the extra circle, the theory comes back to itself transformed by a duality transformation. In the full string theory or M-theory, the duality symmetry is broken to a discrete subgroup $K(\mathbb{Z})$ [9] and for the reduction to lift to string or M-theory, the monodromy must be in $K(\mathbb{Z})$ [10]. A subgroup $GL(d; \mathbb{Z})$ of $K(\mathbb{Z})$ acts geometrically as diffeomorphisms of the d -torus, and if the monodromy M is in $GL(d; \mathbb{Z})$, the reduction can be thought of as compactification on a $d + 1$ dimensional space \mathcal{X} which is a T^d bundle over a circle with monodromy M [3, 10]. Moreover, as we shall review in section 2, such a space \mathcal{X} is in fact a twisted torus in the previous sense, i.e. it is locally a group manifold, and is of the form $\mathcal{X} = G/\Gamma$ [3]. In the case of reduction of pure gravity, the group G is precisely the gauge group of the reduced theory.

Here we will focus on the T-duality subgroup $O(d, d; \mathbb{Z}) \subseteq K(\mathbb{Z})$. There is a geometric subgroup of $O(d, d; \mathbb{Z})$ acting through torus diffeomorphisms and integral shifts of the B -field. If the monodromy is not in this subgroup, it is not a compactification, but can be thought of as string theory with a non-geometric internal space, known as a T-fold [6]. However, as $O(d, d; \mathbb{Z}) \subset GL(2d; \mathbb{Z})$, it has a natural action as diffeomorphisms of a ‘doubled torus’ T^{2d} and there is a T^{2d} bundle over a circle with such a monodromy. A formulation using a circle coordinate and its dual arises naturally in the string field theory for toroidal backgrounds [11]. String theory reduced in this way with a duality twist can be formulated as a sigma-model on a bundle over a circle whose fibres are the doubled torus T^{2d} [6]. The doubled formalism has the virtue that it provides a geometric interpretation to many nongeometric backgrounds. The doubled torus has coordinates conjugate to both the d momenta and the d winding numbers. Different dual backgrounds arise from choosing different polarisations or choices of $T^d \subset T^{2d}$, specifying the ‘real’ spacetime slice of the doubled space. T-duality acts to change the choice of polarisation, and T-folds arise when there is no global polarisation.

Such reductions with duality twists give theories with a gauge group of dimension $2(d+1)$, the same as it would be for reduction on an untwisted $S^1 \times T^d$. In that case, the group is $U(1)^{2(d+1)}$ with $U(1)^{d+1}$ from the natural geometric action on $S^1 \times T^d$ and another $U(1)^{d+1}$ from B -field gauge transformations. One might expect that reductions with a duality twist would give a gauge group containing $U(1)^{2d}$ associated with the T^d fibres. As it happens, this is not the case, and we give a careful derivation of the gauge algebra here. One of the aims of this paper is to explore the implications of the structure of the gauge algebra.

In the doubled formalism, the T^d fibre is doubled, and this raises the question of whether the base S^1 might also be doubled. This would be relevant for the issue of whether one can T-dualise over the base circle. As the geometry has non-trivial dependence on the S^1 coordinate x , there is no isometry on the circle so the usual formulations of T-duality do

not apply. However, in [12] a generalisation of T-duality to such situations was proposed, in which dependence on the circle coordinate x is transformed under T-duality to dependence on the coordinate of a dual circle, \tilde{x} . In this context it is natural to consider more general reductions involving independent duality twists over x and \tilde{x} . Such backgrounds would not admit a geometric description even locally.

Conventional considerations are insufficient to discuss the situation with non-trivial dependence on \tilde{x} . Here, we identify a $2(d+1)$ dimensional doubled geometry that extends the doubled torus bundle to include one other dimension, with coordinate \tilde{x} , and which is the group manifold of the gauge group, identified under a discrete subgroup to give a doubled twisted torus. This is the natural space to include all possible dual backgrounds, including the ones involving the conjectured generalised T-duality on the base S^1 . The full gauge group contains generators acting geometrically on the original space and ones acting as B -field gauge transformations, while in this doubled picture, all arise geometrically.

The canonical example of the types of string background discussed above is given by a sequence of T-dualities starting from the three-dimensional nilmanifold. This nilmanifold is a T^2 bundle over S^1 with monodromy given by a parabolic element of $SL(2; \mathbb{Z})$. T-duality interchanges various quantities referred to as generalised fluxes in [13] and called f -flux, Q -flux and R -flux in [13]. As will be reviewed in the next section, the nilmanifold may be thought of as a twisted torus characterized by the structure constant (or ‘geometric flux’) $f_{xz}{}^y = m \in \mathbb{Z}$ [10, 3]. The Buscher rules can be applied fibrewise to dualise along the T^2 fibre directions. Dualising along the y direction gives a T^3 with H -flux $K_{xyz} = m$, whilst dualising along the z direction gives a T-fold, characterized by the ‘flux’ $Q_x{}^{zy} = m$. It was conjectured in [12] that a further T-duality along the x -direction gives rise to a background constructed as a T^2 fibration over the dual coordinate \tilde{x} with a T-duality twist. This conjectured background is characterized by the nongeometric flux $R^{xyz} = m$ (or ‘R-flux’). The duality sequence may be summarized as [13]

$$K_{xyz} \rightarrow f_{xz}{}^y \rightarrow Q_x{}^{yz} \rightarrow R^{xyz} \quad (1.2)$$

Whilst the doubled formalism has been successfully employed to give a geometric description of the T-fold, such an understanding of the backgrounds with R -flux has not been forthcoming. It is the aim of this paper to shed some light on the group theoretic and geometric structures which underly the duality sequence above. In particular we shall see that a knowledge of the gauge algebra of the compactified theory suggests a natural local structure for a doubled internal space.

A key objective is to understand how to lift a general gauged supergravity to superstring theory. The structure constants of the gauge algebra can be thought of as arising from the various types of flux, and so this is a question of understanding backgrounds with f, H, Q or R fluxes, and in particular the non-geometric ones with Q or R flux.

The plan of this paper is as follows: In the next section we review the relationship

between duality twist backgrounds with geometric monodromy and twisted tori. Section 3 will consider the general $O(d, d)$ -twisted reduction. In section 4 the Yang-Mills gauge symmetries of this theory will be studied. Section 5 describes the doubled geometry of the backgrounds considered here.

2 Reductions with a Geometric Duality Twist

In this section we review the Scherk-Schwarz reduction with a geometric twist [3] and its relation to a reduction on a twisted torus of the form G/Γ .

Consider a $D + d + 1$ dimensional field theory. We reduce the theory on a d -dimensional torus T^d , with real coordinates $z^a \sim z^a + 1$ where $a = 1, 2 \dots d$. This produces a theory in $D + 1$ dimensions with scalar fields that include those in the coset $GL(d; \mathbb{R})/SO(d)$ arising from the torus moduli. Truncating to the z^a independent zero mode sector gives a theory that has a rigid $GL(d; \mathbb{R})$ symmetry, while in the full Kaluza-Klein theory this is broken to $GL(d; \mathbb{Z})$ – the mapping class group of the T^d . Let

$$ds^2 = \hat{G}_{ab} dz^a dz^b \quad (2.1)$$

be the metric on the d -torus. The symmetric matrix \hat{G}_{ab} parameterises the moduli space $GL(d; \mathbb{R})/SO(d)$. There is a natural action of $GL(d; \mathbb{R})$ on the metric and coordinates z^a in which

$$\hat{G}_{ab} \rightarrow (U^t)_a{}^c \hat{G}_{cd} U^d{}_b \quad z^a \rightarrow (U^{-1})_b{}^a z^b \quad (2.2)$$

where $U^b{}_a \in GL(d, \mathbb{R})$. We now truncate to a massless $D + 1$ dimensional field theory and consider reduction on a further circle. In the twisted reduction, dependence on the circle coordinate x is introduced through a $GL(d; \mathbb{R})$ transformation $U = \gamma(x)$ where $\gamma(x) = \exp(Nx)$ and $N^a{}_b$ is some matrix in the Lie algebra of $GL(d; \mathbb{R})$. This defines the x -dependence of the torus moduli through

$$G(x)_{ab} = (\gamma(x)^t)_a{}^c \hat{G}_{cd} \gamma(x)^d{}_b \quad (2.3)$$

for some arbitrary choice \hat{G}_{ab} . The monodromy round the circle $x \sim x + 1$ is $e^N \in GL(d; \mathbb{R})$. The truncation of all Kaluza-Klein modes gives the Scherk-Schwarz reduction [3]. A necessary condition for this to lift to a compactification of the original $D + d + 1$ dimensional theory, keeping all Kaluza-Klein modes, is that the monodromy is in $GL(d; \mathbb{Z})$, which puts strong constraints on the choice of N [14]. Assuming $e^N \in GL(d; \mathbb{Z})$, the twisted reduction is equivalent to the reduction on a T^d bundle over S^1 with metric

$$ds_{d+1}^2 = dx^2 + G(x)_{ab} dz^a dz^b = (\sigma^x)^2 + \hat{G}_{ab} \sigma^a \sigma^b \quad (2.4)$$

where

$$\sigma^x = dx \quad \sigma(x)^a = \gamma(x)_b{}^a dz^b \quad (2.5)$$