Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

CORINNA KLAPPROTH, PETER DEUFLHARD, ANTON SCHIELA

A Perturbation Result for Dynamical Contact Problems [1]

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A Perturbation Result for Dynamical Contact Problems†

Corinna Klapproth, Peter Deuflhard, Anton Schiela

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Abstract

This paper is intended to be a first step towards the continuous dependence of dynamical contact problems on the initial data as well as the uniqueness of a solution. Moreover, it provides the basis for a proof of the convergence of popular time integration schemes as the Newmark method.

We study a frictionless dynamical contact problem between both linearly elastic and viscoelastic bodies which is formulated via the Signorini contact conditions. For viscoelastic materials fulfilling the Kelvin-Voigt constitutive law, we find a characterization of the class of problems which satisfy a perturbation result in a non-trivial mix of norms in function space. This characterization is given in the form of a stability condition on the contact stresses at the contact boundaries.

Furthermore, we present perturbation results for two well-established approximations of the classical Signorini condition: The Signorini condition formulated in velocities and the model of normal compliance, both satisfying even a sharper version of our stability condition.

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1 Introduction

One of the most popular time discretization schemes for dynamical contact problems is the Newmark method. Unfortunately, this scheme may lead to artificial numerical oscillations at dynamical contact boundaries and an undesirable energy blow-up during time integration may occur [5, 20]. In [17], Kane et al. suggested a variant of Newmark's method which is energy dissipative at contact. Unfortunately, this scheme is still unable to circumvent the undesirable oscillations at contact boundaries, for which reason Deuflhard et al. suggested a contact-stabilized Newmark method [5, 20]. Up to now, the question of convergence of Newmark schemes in the

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presence of contact has completely been avoided in both the engineering and the mathematical literature – a difficult problem due to the non-smoothness at contact boundaries. Our aim is to apply the established proof technique for discretizations of evolution problems by Hairer et al. (also known as "Lady Windermere's Fan", cf. [11]) to the contact-stabilized Newmark method. For this purpose, the necessary first step is to find a norm in which we can expect a perturbation result even in the presence of contact.

The classical approach modeling contact phenomena between elastic bodies employs Signorini's contact conditions which are based on the non-penetrability of mass and lead to nonsmooth and nonlinear variational inequalities. The first existence and uniqueness results for evolution problems in elasticity were obtained by Duvaut and Lions [6]. They studied the special case of prescribed normal stresses where the contact surface is known in advance. For the case of linear elasticity in conjunction with Signorini's contact conditions, to date, existence has only been provided in some simple geometric settings and for one-dimensional problems. A general theory for multi-dimensional dynamical contact problems is still missing.

Basically, the serious mathematical difficulties with the well-posedness of purely elastic problems result from the irregularity of the velocities at contact. However, the assumption of viscous material behavior allows the derivation of existence results. In [13] and [15], Jarušek proved the existence of a weak solution for the dynamical frictionless Signorini problem between a viscoelastic body with singular memory and a rigid foundation. More recently, the existence of weak solutions in viscoelasticity with Kelvin-Voigt constitutive law has been studied by Cocou [3] and Kuttler and Shillor [23]. Migòrski and Ochal [27] established the existence for a class of unilateral viscoelastic contact problems modeled by dynamical hemivariational inequalities. In 2008, Ahn and Stewart [1] proved an existence result for a frictionless dynamical contact problem between a linearly viscoelastic material of Kelvin-Voigt type and a rigid obstacle. A survey of existence and uniqueness results is given in the monograph [9].

Unfortunately, there are still fundamental and unresolved mathematical difficulties even in the analysis of viscoelastic contact. These are caused by the Signorini conditions on the unknown displacement field itself. Therefore, Jarušek and Eck investigated the solvability of dynamical contact problems with unilateral contact constraints on the velocity field (cf. [14, 16]). This approach yields the monotonicity of the corresponding multivalued operator.

Martins and Oden [26] proposed the normal compliance condition, likewise leading to a problem with a much simpler mathematical structure. Their model assumes that the normal stresses on the contact surface depend only on the normal displacement field which results in a relaxation of the non-penetration of mass. They presented existence and uniqueness results for linearly elastic and viscoelastic materials, but unfortunately their proof of uniqueness exhibits a fundamental error in the estimation of norms. Their model of normal compliance was used in various papers, see, e.g., [4, 19, 21, 22] and the monograph [18]. One of its main advantages is the higher regularity of the solutions in time [24]. However, for the medical applications that we have in mind (such as the movement of the knee joint, see [20]), a mutual interpenetration of the bodies is unacceptable and normal compliance models are ruled out.

Most of the papers cited above concern existence and uniqueness results for dynamical contact problems. However, to the best of our knowledge, there are still no mathematical results concerning the continuous dependence on the initial data. The lack of well-posedness results mainly originates from the hyperbolic structure of the problem which leads to shocks at the contact interfaces. The Signorini conditions in displacements seem to avoid a general regularity of such problems.

The paper is organized as follows. In Section 2, we will consider the frictionless dynamical contact problem between two linearly elastic bodies based on Signorini's conditions. We will give a short description of the underlying physical and mathematical model. Then, we will point out the essential mathematical difficulties in the derivation of a perturbation result for such materials. To this end, in Section 3, we will introduce the Kelvin-Voigt model for viscoelastic materials. We will find a characterization of a class of problems for which it is possible to prove a perturbation result in a non-trivial choice of mixed norms in function space. In Section 4, we will end with two famous approximations of the Signorini condition in linear viscoelasticity, namely the Signorini conditions in velocities and the normal compliance model. For these, we will give perturbation results yielding even the uniqueness of the approximated solutions.

2 The Signorini condition in linear elasticity

The first two sections of this paper deal with a perturbation result for dynamical contact problems with Signorini conditions in displacements. To the best of our knowledge, there exist no results concerning continuous dependence on the initial data in the mathematical literature, neither in the purely elastic nor in the viscoelastic case.

In Section 2.1, we will give a short description of the classical contact problem in linear elasticity which is formulated via the Signorini conditions. Then, in Section 2.2, we will analyze the fundamental problems with a characterization of linearly elastic contact problems which are satisfying a perturbation result.

2.1 Theoretical Background

We use the same model of dynamical contact between two linearly elastic bodies as in [5] which is based on Signorini's contact conditions. For the convenience of the reader, we will briefly present the notation of the paper and the formulation of the underlying mathematical model.

Notation. All domains treated here are understood to be bounded subsets in $\mathbb{R}^d$ with $d = 2, 3$ and indices i, j, l, m run from 1 to d throughout the paper. Let