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Barrier Methods for Optimal Control Problems with Convex Nonlinear Gradient Constraints †

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Abstract

In this paper we are concerned with the application of interior point methods in function space to gradient constrained optimal control problems, governed by partial differential equations. We will derive existence of solutions together with first order optimality conditions. Afterwards we show continuity of the central path, together with convergence rates depending on the interior point parameter.

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1 Introduction

In a large number of processes that are modeled using partial differential equations bounds on the gradient of the state variable are of vital importance for the underlying model: large temperature gradients during cooling or heating processes may lead to destruction of the object, that is being cooled or heated; in elasticity the gradient of the deformation determines the change between elastic and plastic material behavior. In any attempt to optimize such processes the gradient therefore has to be regarded. However, not much attention was given to constraints of gradient type, see [4–7, 11, 25]

Problems with constraints on the state (pointwise or regarding the gradient) form a class of highly nonlinear and non-smooth problems. A popular approach for their efficient solution are path-following methods, which solve a sequence of easier to tackle problems. These methods are constructed in a way such that the sequence of the solutions converges to the solution of the original problem. Among these

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methods one can distinguish three main lines of research. Lavrentiev regularization methods due to Tröltzsch et al. [8,18,19,24], Moreau-Yoshida approximation methods due to Hintermüller and Kunisch [1,2,16,17] and interior point methods [22,23]. While the first two candidates abandon feasibility to improve the regularity of the dual variables, interior point methods yield feasible solutions and aim towards smooth systems of equations.

Application of interior point methods to gradient bounds has been proposed in [25] together with a posteriori error estimates with respect to the interior point parameter and the discretization error.

In this paper we perform the analysis of the homotopy path generated by barrier methods to problems with gradient bounds. We approach this problem on the base of the analysis in [23], where pointwise state constraints are considered. Although we can build up on techniques and results established there, it will turn out that a number of interesting, additional issues arise in the case of gradient bounds. For example, the topological framework has to be chosen differently with a C^1-norm, and in contrast to pointwise state constraints the gradient bounds considered here are nonlinear.

Our paper is structured as follows. In Section 2 we establish an abstract theoretical framework for our analysis and illustrate the application of the framework to some PDE constrained optimal control problems. In Section 3 we consider barrier functionals for gradient bounds and characterize their subdifferentials. Then existence of minimizers and first order optimality conditions are established, together with uniform bounds on the barrier gradients. Finally we consider the convergence of the path of minimizers and derive an order of convergence for a typical case.

2 Gradient Constrained Optimal Control Problems

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $\emptyset \neq \Omega_C \subseteq \Omega$ be an open subset, and let $\overline{\Omega}_C$ be its closure. Define the space of states U as a closed subspace of $C^1(\overline{\Omega}_C) \times L^2(\Omega \setminus \overline{\Omega}_C)$, which is clearly a Banach space, and let $W \subset U$ be a dense subspace of U. Consider $W = W^{2,p}(\Omega) \subset U = C^1(\overline{\Omega}_C) \times L^2(\Omega \setminus \overline{\Omega}_C)$ with $p > d$ for an example.

Further, consider two reflexive Banach spaces Q and Z, which will denote the space of controls and the space for the adjoint state, respectively. We denote the corresponding dual spaces by U^*, Q^*, and Z^*. Consider the following abstract linear partial differential equation on Ω:

$$Au = Bq \tag{2.1}$$

where we require the following properties:

Assumption 1. *Assume that $A : U \supset \operatorname{dom} A = W \to Z^*$ is densely defined and possesses a bounded inverse. Further let $B : Q \to Z^*$ be a continuous operator.*

We will see later that continuous invertibility of A is equivalent to closedness and bijectivity. The distinction between the state space U and the domain of definition W of A allows us to consider our optimal control problem in a convenient topological framework (the topology of U), while being able to model differential operators by A, which are only defined on a dense subspace W.

To define an optimal control problem, we specify an objective functional J with some basic regularity assumptions:

Assumption 2. *Let $J = J_1 + J_2$. We assume that $J_1 : U \to \mathbb{R}$ and $J_2 : Q \to \mathbb{R}$ are lower semi-continuous, convex and Gâteaux differentiable. In addition let J_1 be bounded from below and J_2 be strictly convex. Assume that the derivatives are uniformly bounded on bounded sets. This means that there exists a continuous $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|J_1'(u)\|_{U^*} \leq g(\|u\|_U)$ and $\|J_2'(q)\|_{Q^*} \leq g(\|q\|_Q)$.*

We now consider the following minimization problem

$$\min_{Q^{\text{ad}} \times W} \; J(q,u) = J_1(u) + J_2(q), \tag{2.2a}$$

$$\text{s.t.} \quad Au = Bq, \tag{2.2b}$$

$$\text{and} \quad |\nabla u(x)|^2 \leq \psi(x) \text{ on } \overline{\Omega}_C \tag{2.2c}$$

where $\psi \in C(\overline{\Omega}_C)$ with $\psi \geq \delta > 0$ and $Q^{\text{ad}} \subset Q$ closed and convex.

In order to ensure that there exists a solution we require that the following assumption holds

Assumption 3. *We assume that at least one of the following holds:*

(1) *Q^{ad} is bounded in Q.*

(2) *J_2 is coercive on Q.*

For the discussion of interior point methods for the gradient constraint we have to require an additional property, which is of Slater type

Assumption 4. *Assume there exists a feasible control $\check{q} \in Q^{ad}$, such that the corresponding state $\check{u}$ given by $A\check{u} = B\check{q}$ is strictly feasible, that is, $|\nabla \check{u}|^2 < \psi$.*

Lemma 2.1. *Let U be a Banach space. An operator $A : U \supset W \to Z^*$ has a continuous inverse if and only if A is closed and bijective.*

If Assumption 1 holds, then there exists a continuous "control-to-state" mapping

$$S : Q \to U, \quad S := A^{-1}B.$$

Proof. For our first assertion, cf. [22]. By Assumption 1 both A^{-1} and B exist and are continuous, and thus $S := A^{-1}B$, too. □