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Hyperdeterminants as integrable discrete systems

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Abstract

We give the basic definitions and some theoretical results about hyperdeterminants, introduced by A. Cayley in 1845. We prove integrability (understood as $4d$-consistency) of a nonlinear difference equation defined by the $2\times2\times2$ - hyperdeterminant. This result gives rise to the following hypothesis: the difference equations defined by hyperdeterminants of any size are integrable.

We show that this hypothesis already fails in the case of the $2\times2\times2\times2$ - hyperdeterminant.

1 Introduction

Discrete integrable equations have become a very vivid topic in the last decade. A number of important results on the classification of different classes of such equations, based on the notion of consistency [3], were obtained in [1, 2, 17] (cf. also references to earlier publications given there). As a rule, discrete equations describe relations on the scalar field variables $f_{i_1\ldots i_n} \in \mathbb{C}$ associated with the points of a lattice $\mathbb{Z}^n$ with vertices at integer points in the n-dimensional space $\mathbb{R}^n = \{(x_1, \ldots, x_n) | x_s \in \mathbb{R}\}$. If we take the elementary cubic cell $K_n = \{(i_1, \ldots, i_n) \mid i_s \in \{0, 1\}\}$ of this lattice and the field variables $f_{i_1\ldots i_n}$ associated to its 2^n vertices, an n-dimensional discrete system of the type considered here is given by an equation of the form

$$Q_n(\mathbf{f}) = 0. \tag{1}$$

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Hereafter we use the short notation $\mathbf{f}$ for the set $(f_{00\ldots0}, \ldots, f_{11\ldots1})$ of all these 2^n variables. For the other elementary cubic cells of $\mathbb{Z}^n$ the equation is the same, after shifting the indices of $\mathbf{f}$ suitably.

The equations mostly investigated so far [1, 2, 17] were supposed to have the following properties:

1) Quasilinearity. Equation (1) is affine linear w.r.t. every $f_{i_1 i_2 \ldots i_n}$, i.e. Q has degree 1 in any of its four variables.

2) Symmetry. Equation (1) should be invariant w.r.t. the symmetry group of elementary cubic cell K_n or its suitably chosen subgroup.

On the other hand a number of interesting discrete equations which do not enjoy one or both of these properties has been found. In this publication we investigate an important class of symmetric discrete equations which do not have the quasilinearity property and are given by the equations $H_n(f_{00\ldots0}, \ldots, f_{11\ldots1}) = 0$, were H_n denotes the n-dimensional hyperdeterminant of the corresponding n-index array $(f_{00\ldots0}, \ldots, f_{11\ldots1})$. We give the precise definition of hyperdeterminants in Section 2. In the simplest two-dimensional case of the 2×2 matrix the hyperdeterminant is just the usual determinant:

$$H_2(\mathbf{f}) = f_{00}f_{11} - f_{01}f_{10}. \tag{2}$$

The next nontrivial case is the 3-dimensional $2 \times 2 \times 2$ - hyperdeterminant:

$$\begin{aligned} H_3(\mathbf{f}) = \; & f_{111}^2 f_{000}^2 + f_{100}^2 f_{011}^2 + f_{101}^2 f_{010}^2 + f_{110}^2 f_{001}^2 \\ & -2f_{111}f_{110}f_{001}f_{000} - 2f_{111}f_{101}f_{010}f_{000} \\ & -2f_{111}f_{100}f_{011}f_{000} - 2f_{110}f_{101}f_{010}f_{001} \\ & -2f_{110}f_{100}f_{011}f_{001} - 2f_{101}f_{100}f_{011}f_{010} \\ & +4f_{111}f_{100}f_{010}f_{001} + 4f_{110}f_{101}f_{011}f_{000}. \end{aligned} \tag{3}$$

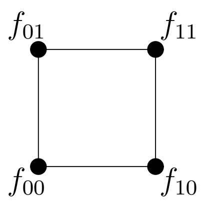

Figure 1: Square K_2.

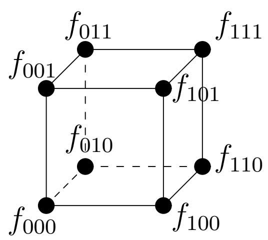

Figure 2: Cube K_3.

The corresponding elementary cells K_2, K_3 and the field variables associated with the vertices are shown on Figures 1, 2.

The general definition of hyperdeterminants was given by A. Cayley [7], who also gave the explicit form (3) of the first nontrivial $2\times2\times2$ - hyperdeterminant. In the last decades, following the modern and much more general approach of $\mathcal{A}$-discriminants [9], the theory of hyperdeterminants found important applications in quantum informatics [4], biomathematics [5], numerical analysis and data analysis [6] as well as other fields.

As one can easily see, the expressions (3.7) in [15] and (6.11) in [16], describing some discrete integrable equations, are nothing but the classical Cayley's $2 \times 2 \times 2$ -

hyperdeterminant (3). We prove below in Section 3 that (3) is also integrable in the sense of $(n+1)$-*dimensional consistency* [3]:

An n-dimensional discrete equation (1) is called consistent, if it may be imposed in a consistent way on all n-dimensional faces of a $(n+1)$-dimensional cube.

We give the accurate formulation of this general consistency principle for the case of non-quasilinear expressions similar to (3) in Section 3. For the two-dimensional determinant (2) (which is quasilinear) consistency can be established by a trivial computation; the equation $H_2(f_{00}, f_{11}, f_{01}, f_{10}) = 0$ is obviously linearized by the substitution $f_{ij} = \exp \tilde{f}_{ij}$. Using a result on Principal Minor Assignment Problem proved in [14] we establish $4d$-consistency of the $2 \times 2 \times 2$ - hyperdeterminant (3) in Section 3, cf. Theorem 2 below for the precise formulation.

This result gives rise to the following *Conjecture*: the difference equations defined by hyperdeterminants of any size are integrable in the sense of $(n+1)$-dimensional consistency. Nevertheless as we show in Section 4, this Conjecture fails already in the case of the $2\times2\times2\times2$ - hyperdeterminant. The computation of this $4d$ - hyperdeterminant turns out to be highly nontrivial (compared to the relatively simple expressions (2), (3)) and was completed only recently [13]. We report in Section 4 a more straightforward and simpler computation of the same hyperdeterminant with the free symbolic computation program FORM [18]. The size of this hyperdeterminant (2 894 276 terms, total degree 24, degree 9 w.r.t. each of the field variables) implies that checking its $5d$-consistency can be done only numerically, using high precision computation of roots of respective polynomial equations on the $4d$-faces of the 5-dimensional cube K_5. This was done using two different computer algebra systems REDUCE [19] and SINGULAR [20]. As our computations have shown (cf. their description in Section 4), the $4d$ - hyperdeterminantal equation $H_4(\mathbf{f}) = 0$ is not $5d$-consistent. This non-integrability result should be investigated further since recent examples [11] show that consistency is not the only possible definition for discrete integrability.

2 The definition of hyperdeterminants and its variations

The remarkable definition of hyperdeterminants given by A. Cayley in 1845 [7] and still used today [9] describes the condition of singularity of an appropriate hypersurface. Let $A = (a_{i_1 i_2 \cdots i_r})$ be a hypermatrix (an array with r indices) with $i_s = 0, \ldots, n_s$. The polylinear form

$$U = \sum_{i_1 \cdots i_r} a_{i_1 \cdots i_r} x^{(1)}_{i_1} \cdots x^{(r)}_{i_r}$$

defines a hypersurface $U = 0$ in $\mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_r}$. Here $x^{(k)}_{i_k}$ denote the homogeneous coordinates in the respective complex projective space $\mathbb{C}P^{n_k}$. This hypersurface is singular, i.e. has at least one point where the condition of smoothness is not satisfied iff the following set of $(n_1+1)\cdot\ldots\cdot(n_r+1)$ equations

$$\left\{\forall s=1,\ldots,r, \quad \forall k=1,\ldots,n_s, \quad \frac{\partial U}{\partial x_{i_s}^{(k)}}=0\right\} \tag{4}$$

has a nontrivial solution $x_{i_s}^{(k)} \in \mathbb{C}P^{n_k}$. As one can show (cf. [9]), if a certain condition (5) on the dimensions n_k of the array A is satisfied, elimination of the variables $x_{i_s}^{(k)}$ from (4) results in a single polynomial equation in the array elements $a_{i_1 i_2 \cdots i_r}$: $H_r(A)=0$. This polynomial is irreducible and enjoys practically the same symmetry properties as the usual determinant of a square matrix. Following Cayley this polynomial $H_r(A)$ (defined uniquely up to a constant factor) is called the *hyperdeterminant of the array A*. The necessary and sufficient condition of existence of a *single* polynomial condition $H_r(A)=0$ for the hypersurface $U=0$ to be singular, i.e. the condition for the corresponding hyperdeterminant of A to be correctly defined, is as follows:

$$\forall k, \qquad n_k \le \sum_{s\neq k} n_s. \tag{5}$$

In particular, if $r=2$, so for usual $(n_1+1)\times(n_2+1)$-matrices, this condition implies $n_1=n_2$, and in this case the hyperdeterminant H_2 coincides with the classical determinant of the matrix $A_{i_1 i_2}$. Note that for a given set $\{n_1,\ldots,n_r\}$ of array dimensions one says that we have the corresponding $(n_1+1)\times\ldots\times(n_r+1)$ - hyperdeterminant since the array indices range from 0 to n_k. The hyperdeterminant is $SL(\mathbb{C},n_1+1)\times\cdots\times SL(\mathbb{C},n_r+1)$-invariant, which means that if one adds to one *slice* $A_{k,p}=\{(a_{i_1 i_2\cdots i_r})\ \text{with fixed}\ i_k=p\}$ another parallel slice $A_{k,q}$, $q\neq p$, multiplied by some constant λ, the value of H_r is unchanged; swapping the slices $A_{k,p}$, $A_{k,q}$ either leave H_r again invariant or changes its sign depending on the parity of the dimensions n_i; finally, multiplication of a slice $A_{k,p}$ with a constant λ results in multiplication of the hyperdeterminant by an appropriate power of λ. H_r is also invariant w.r.t. the transposition of any two indices i_l, i_m of the hypermatrix $A=(a_{i_1 i_2\cdots i_r})$.

As we have stated in the introduction, the first nontrivial $2\times2\times2$ - hyperdeterminant (3) was computed by A. Cayley himself [7]. Amazingly enough, already the next step, computation of the $2\times2\times2\times2$ - hyperdeterminant is very difficult. The problem of computation of an *explicit* polynomial expression for this case was proposed by I. M. Gel'fand in his Fall 2005 research seminar at Rutgers University. The monomial expansion of the $2\times2\times2\times2$ - hyperdeterminant is related to some combinatorial problems, and was done (using an inductive algorithm of L. Schläfli [8]) for the first time in [13], using a dedicated C code; this computation required a serious programming effort since the standard computer algebra systems like Maple can not cope with the intermediate large expressions. The resulting polynomial expression for the $2\times2\times2\times2$ - hyperdeterminant has 2 894 276 terms, total degree 24, and has degree 9 w.r.t. each of the array entries $a_{i_1 i_2 i_3 i_4}$. The size of this expression in usual text format is around 200 megabytes.

In October 2007 we re-checked this computation of the $2\times2\times2\times2$ - hyperdeterminant using a free symbolic computation program Form [18] and the same inductive