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Approximations for the second moments of sojourn times in M/GI systems under state-dependent processor sharing

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Abstract

We consider a system with Poisson arrivals and general service times, where the requests are served according to the State-Dependent Processor Sharing (SDPS) discipline (Cohen's generalized processor sharing discipline), where each request receives a service capacity which depends on the actual number of requests in the system. For this system, denoted by $M/GI/SDPS$, we derive approximations for the squared coefficients of variation of the conditional sojourn time of a request given its service time and of the unconditional sojourn time by means of two-moment fittings of the service times. The approximations are given in terms of the squared coefficients of variation of the conditional and unconditional sojourn time in related $M/D/SDPS$ and $M/M/SDPS$ systems, respectively. The numerical results presented for $M/GI/m{-}PS$ systems illustrate that the proposed approximations work well.

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1 Introduction

Processor Sharing (PS) systems have been widely used in the last decades for modeling and analyzing computer and communication systems, cf. e.g.

[BBJ], [BB2], [GRZ], [Ott], [PG], [Yas], and the references therein. In this paper we deal with approximations for the second moment of the conditional sojourn time $V(\tau)$ of a request with required service time τ (τ-request) and of the unconditional sojourn time V in the following system, denoted by $M/GI/SDPS$: At a node requests arrive according to a Poisson process of intensity λ with i.i.d. service times, which are independent of the arrival process and have the df. $B(x) := P(S \le x)$ with finite mean $m_S := ES$, where S denotes a generic service time. The requests are served according to the following State-Dependent Processor Sharing (SDPS) discipline, cf. [Coh], [BBJ]: If there are $n \in \mathbb{N} := \{1, 2, \ldots\}$ requests in the node then each of them receives a positive service capacity $\varphi(n)$, i.e., each of the n requests receives during an interval of length $\Delta\tau$ the amount $\varphi(n)\Delta\tau$ of service. In case of $\varphi_1(n) = 1/n$, $n \in \mathbb{N}$, we obtain the well known $M/GI/1 - PS$ system, cf. e.g. [Yas]. In case of $\varphi_{1,k}(n) = 1/(n + k)$, $n \in \mathbb{N}$, we have a single-server PS system with $k \in \mathbb{N}$ permanent requests in the system, in case of $\varphi_m(n) = \min(m/n, 1)$, $n \in \mathbb{N}$, an $M/GI/m - PS$ system, i.e. an m-server PS system, where all requests are served in a PS mode, but each request receives at most the capacity of one processor, cf. [Coh] p. 283, [Bra], [GRZ]. Finally, in case of $\varphi(n) = 1$, $n \in \mathbb{N}$, we have an $M/GI/\infty$ system.

The $M/GI/SDPS$ system is stable if and only if

$$\sum_{n=0}^{\infty} \prod_{j=1}^{n} \frac{\varrho}{j\varphi(j)} < \infty, \tag{1.1}$$

where $\varrho := \lambda m_S$ denotes the offered load, cf. [Coh] (7.18). We assume in the following that the system is stable and in steady state. Then the distribution $p(n) := P(N = n)$, $n \in \mathbb{Z}_+$, of the stationary number N of requests in the system is given by, cf. [Coh] (7.19),

$$p(n) = \left(\sum_{m=0}^{\infty} \prod_{j=1}^{m} \frac{\varrho}{j\varphi(j)} \right)^{-1} \prod_{j=1}^{n} \frac{\varrho}{j\varphi(j)}, \quad n \in \mathbb{Z}_+. \tag{1.2}$$

For the unconditional sojourn time V of an arriving request from Little's law and (1.2) we find that

$$EV = \frac{1}{\lambda} \sum_{n=0}^{\infty} np(n) = m_S \sum_{n=0}^{\infty} \frac{1}{\varphi(n+1)}\, p(n), \tag{1.3}$$

and for the conditional sojourn time $V(\tau)$ of a τ-request it holds

$$EV(\tau) = \frac{\tau}{m_S}\, EV = \tau \sum_{n=0}^{\infty} \frac{1}{\varphi(n+1)}\, p(n), \quad \tau \in \mathbb{R}_+, \tag{1.4}$$

cf. [Coh] (7.27). More generally, for $\tau \in \mathbb{R}_+$, $k \in \mathbb{N}$ we have the estimate

$$\tau^k \Big(\sum_{n=0}^{\infty} \frac{1}{\varphi(n+1)}\, p(n) \Big)^k \leq E[V^k(\tau)] \leq \tau^k \sum_{n=0}^{\infty} \Big(\frac{1}{\varphi(n+1)} \Big)^k p(n), \quad (1.5)$$

and it holds

$$\lim_{\tau \downarrow 0} \frac{E[V^k(\tau)]}{\tau^k} = \sum_{n=0}^{\infty} \Big(\frac{1}{\varphi(n+1)} \Big)^k p(n), \quad (1.6)$$

cf. [BB3] Theorem 3.1. It seems that for the general $M/GI/SDPS$ system for V and $V(\tau)$ besides (1.3)–(1.6) there are known only asymptotic results for heavy tailed service times, cf. [GRZ]. However, for special cases several results and numerical algorithms are known. For the $M/GI/1-PS$ system and special cases, cf. e.g. [Yas], [SGB], for the $M/M/m-PS$ system cf. [Bra], for the general $M/M/SDPS$ system cf. [BB2]. For the $M/GI/SDPS$ system with service times exponentially distributed in a neighborhood of zero as well as for the $M/D/SDPS$ system and in particular for the $M/D/2-PS$ system cf. [BB4]. In [vBe] there are given simple approximations for the second moments of $V(\tau)$ and V in the $M/GI/1-PS$ system. For an approximation of V in the $GI/GI/1-PS$ system see [Sen], for an approximation and an upper bound of EV in the $G/GI/1-PS$ system see [BB1].

The aim of this paper is to derive for the $M/GI/SDPS$ system in Section 2 approximations for the second moment of $V(\tau)$ and in Section 3 approximations for the second moment of V. They are based on two two-moment fittings of the service times, and are given in terms of the first two moments of $V(\tau)$ and V in related $M/D/SDPS$ and $M/M/SDPS$ systems, respectively. The numerical results presented for $M/GI/m-PS$ systems in Section 4 illustrate that the proposed approximations work well and can be computed efficiently.

2 Approximations for the second moment of $V(\tau)$

We assume that the $M/GI/SDPS$ system is stable and in steady state. Further, we assume that $\sum_{n=0}^{\infty} \varphi(n+1)^{-2}\, p(n)$ is finite, ensuring that $E[V^2(\tau)]$ is finite, too, cf. (1.5).

Instead of dealing with the second moment of a r.v. X, we consider its squared coefficient of variation (scv) $c^2(X) := var(X)/(EX)^2$ in the following. Note that from (1.4)–(1.6) it follows that

$$c^2(V(\tau)) \leq c^2(1/\varphi(N+1)), \quad \tau \in (0, \infty), \quad (2.1)$$

with equality for $c^2(V(0)) := \lim_{\tau\downarrow 0} c^2(V(\tau))$. For obtaining approximations for $c^2(V(\tau))$ based on a two-moment characterization of $B(x)$ given by m_S and $c^2(S)$, we model the service time by a mixture of a zero and a positive service time d_0:

$$B_{D,p}(x) := (1-p)\,\mathbb{I}\{x \geq 0\} + p\,\mathbb{I}\{x \geq d_0\}, \quad x \in \mathbb{R}_+, \tag{2.2}$$

where $p \in (0,1]$, $d_0 \in (0,\infty)$. The mean and scv of (2.2) are given by $m_S = p\,d_0$, $c^2(S) = (1-p)/p$. The df. (2.2) can be used for modeling arbitrary service times with given mean $m_S \in (0,\infty)$ and scv $c^2(S) \in [0,\infty)$ since the parameters

$$p := 1/(c^2(S)+1), \qquad d_0 := m_S/p \tag{2.3}$$

provide the desired mean and scv, i.e., (2.2), (2.3) provide a two-moment fitting for arbitrary service times. Assume now that S has the df. $B_{D,p}(x)$. Note that under the $SDPS$ discipline the sojourn times of the zero service time requests are zero and that they do not have any impact on the system dynamics. Thus the dynamics of the $M/GI/SDPS$ system correspond to these of an $M/D/SDPS$ system with arrival intensity $\lambda_0 := p\lambda$ and deterministic service times $d_0 = m_S/p$. Therefore the sojourn time $V^{D,p}(\tau)$ of a τ-request in the $M/GI/SDPS$ system with service time df. (2.2), (2.3) equals the sojourn time of a τ-request in the $M/D/SDPS$ system with arrival intensity λ_0 and deterministic service times d_0 in distribution. Time scaling provides that $V^{D,p}(\tau)$ equals $p^{-1}V^D(p\tau)$ in distribution, where $V^D(\tau)$ denotes the sojourn time of a τ-request in the $M/D/SDPS$ system with arrival intensity λ and deterministic service times $d := m_S$. In particular, we find that

$$c^2(V^{D,p}(\tau)) = c^2(V^D(p\tau)) = c^2(V^D(\tau/(c^2(S)+1))), \tag{2.4}$$

cf. (2.3). Note that (2.4) implies that (2.1) is tight for any $\tau \in (0,\infty)$ with respect to $B(x)$ given m_S by choosing p sufficiently small, i.e., within a one-moment characterization of the service times (2.1) cannot be improved. In case of an arbitrary $M/GI/SDPS$ system with service time df. $B(x)$, characterized by m_S and $c^2(S)$, the r.h.s. of (2.4) provides the following first approximation for $c^2(V(\tau))$:

$$c^2(V(\tau)) \approx c^2_{app1}(V(\tau)) := c^2(V^D(\tau/(c^2(S)+1))). \tag{2.5}$$

The approximation (2.5) bases on approximating $B(x)$ via the two-moment matching (2.2), (2.3). Note that for any approximation of $c^2(V(\tau))$ in $M/GI/SDPS$ given by a two-moment fitting of arbitrary service times,