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G_2 generating technique for minimal $D = 5$ supergravity and black rings

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A solution generating technique is developed for $D = 5$ minimal supergravity with two commuting Killing vectors based on the G_2 U-duality arising in the reduction of the theory to three dimensions. The target space of the corresponding 3-dimensional sigma-model is the coset $G_{2(2)}/(SL(2, R) \times SL(2, R))$. Its isometries constitute the set of solution generating symmetries. These include two electric and two magnetic Harrison transformations with the corresponding two pairs of gauge transformations, three $SL(2, R)$ S-duality transformations, and the three gravitational scale, gauge and Ehlers transformations (altogether 14). We construct a representation of the coset in terms of 7×7 matrices realizing the automorphisms of split octonions. Generating a new solution amounts to transforming the coset matrices by one-parametric subgroups of $G_{2(2)}$ and subsequently solving the dualization equations. Using this formalism we derive a new charged black ring solution with two independent parameters of rotation.

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I. INTRODUCTION

The discovery of rotating black rings [1] (for a recent review see [2] and references therein) has attracted new interest in five-dimensional minimal supergravity [3, 4]. Within this theory supersymmetric charged black ring solutions were found [5, 6]. The bosonic sector of five-dimensional minimal supergravity is Einstein-Maxwell theory with a Chern-Simons term, the structure of the Lagrangian being similar to that of eleven-dimensional supergravity [7, 8]. While in pure Einstein-Maxwell theory in five and higher dimensions no charged black hole solution, generalizing the uncharged Myers-Perry black holes [9], is known, the Chern-Simons term endows five-dimensional Einstein-Maxwell theory with more hidden symmetries, implying the existence of exact charged rotating black hole solutions [10–12], Meanwhile the most general black ring solution which might possess mass, two angular momenta, electric charge and magnetic moment as independent parameters is still not found. Here we propose a new generating technique which can solve this problem. It is based on the duality symmetries of the three-dimensional reduction of the theory.

The hidden symmetries arising upon dimensional reduction of five-dimensional minimal supergravity to three dimensions were studied by Mizoguchi and Ohta [7], by Cremmer, Julia, Lu and Pope [13] using the technique of [14], and were more recently investigated both in the bosonic and fermionic sectors by Possel [15] (see also [16]). The corresponding classical U-duality group is the non-compact version of the lowest rank exceptional group G_2 [17]. In three dimensions one obtains the gravity-coupled sigma-model with the homogeneous target space $G_{2(2)}/SO(4)$ for

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the Lorentzian signature of the 3-space, or $G_{2(2)}/(SL(2,R)\times SL(2,R))$ in the Euclidean case. Some further aspects of these symmetries were discussed in [8], their infinite-dimensional extension upon reduction to two and one dimensions was also explored [18].

Here we investigate the $G_{2(2)}/(SL(2,R)\times SL(2,R))$ sigma model in the context of the solution generating technique which has proved to be extremely useful in various non-linear theories from pure gravity, Einstein-Maxwell theory [19–23] and dilatonic gravity [24–30] to more general supergravity models [31–33] and string theory [34]. Some partial use of hidden symmetries of this kind to generate new rotating rings recently became a rapidly developing industry. One direction was to use the $SL(2,R)$ subgroup of the U-duality group [35, 36]. Another line was related to the purely gravitational sector (without the Maxwell field) which leads to $SL(3,R)$ U-duality in three dimensions [20, 37, 38]. Further reduction to two dimensions gives rise to a Belinsky-Zakharov type integrable model which was extensively used to construct soliton solutions [38–49]. However, the full G_2 symmetry was never used for generating purposes for lack of a convenient representation of the coset $G_{2(2)}/(SL(2,R)\times SL(2,R))$ in terms of the target space variables. Although the 14-dimensional (adjoint) representation was given explicitly in [7], it is still too complicated for practical generating applications. Here we construct a suitable representation in terms of 7×7 matrices and give two examples of its application: a sigma-model construction of the electrically charged rotating black hole and the generation of a non-BPS doubly rotating charged black ring from the black ring with two angular momenta of [47].

Five-dimensional minimal supergravity contains a graviton, two $N=2$ symplectic-Majorana gravitini (equivalent to a single Dirac gravitino), and one $U(1)$ gauge field. The bosonic part of the Lagrangian is very similar to that of $D=11$ supergravity, being endowed with a Chern-Simons term [3, 4]:

$$S_5=\frac{1}{16\pi G_5}\left[\int d^5x\sqrt{-\hat{g}}\left(\hat{R}-\frac{1}{4}\hat{F}^2\right)-\frac{1}{3\sqrt{3}}\int \hat{F}\wedge\hat{F}\wedge\hat{A}\right], \tag{1}$$

where $\hat{F}=d\hat{A}$. This theory can be obtained as a suitably truncated Calabi-Yau compactification of $D=11$ supergravity [50].

Our purpose is to construct a generating technique for classical solutions with two commuting Killing symmetries. Dimensional reduction leads to a three-dimensional sigma-model possessing a $G_{2(2)}$ target space symmetry [7]. To explore it fully we need a convenient representation of the action of symmetries on the target space variables. We give here an alternative derivation of the three-dimensional sigma-model which has the advantage of being more explicit and easy to use for solution generation. The reduction is performed in Sect. 2 in two steps, first to four, then to three dimensions. In Sect. 3 we reveal the symmetries of the three-dimensional sigma-model using a direct (computer assisted) solution of the corresponding Killing equations[1]. The resulting symmetry transformations are identified in the usual terms of gauge, S-duality and Harrison-Ehlers sectors. Then we reformulate in Sect. 4 the problem in terms of a covariant (with respect to the two-Killing plane) reduction which is more suitable for constructing the matrix representation, and give the coset matrix representative as a symmetrical 7×7 matrix. In Sect. 5 we identify the charging transformation, and apply it to the construction of the doubly rotating charged black ring.

II. DIMENSIONAL REDUCTION

A. D=4

Assuming that the five-dimensional metric and the Maxwell field $\hat{A}$ do not depend on a space-like coordinate z, we arrive at the four-dimensional Einstein theory with two Maxwell fields, a dilaton and an axion. We parametrize the five-dimensional interval and the Maxwell one-form as

$$ds_5^2 = \mathrm{e}^{-2\phi}(dz+C_\mu dx^\mu)^2+\mathrm{e}^{\phi}ds_4^2, \tag{2}$$

$$\hat{A} = A_\mu dx^\mu+\sqrt{3}\kappa dz, \tag{3}$$

$(\mu=1\ldots4)$. The corresponding four-dimensional action reads

$$S_4=\frac{1}{16\pi G_4}\int d^4x\sqrt{-g}\left[R-\frac{3}{2}(\partial\phi)^2-\frac{3}{2}\mathrm{e}^{2\phi}(\partial\kappa)^2-\frac{1}{4}\mathrm{e}^{-3\phi}G^2-\frac{1}{4}\mathrm{e}^{-\phi}\tilde{F}^2-\frac{1}{2}\kappa FF^*\right], \tag{4}$$

---

[1] A purely algebraic construction of the Killing vectors will be presented elsewhere [51].

where

$$G_4 = G_5/2\pi R_5, \quad G = dC, \quad F = dA, \quad \tilde{F} = F + \sqrt{3} C \wedge d\kappa, \tag{5}$$

and F^* is the four-dimensional Hodge dual of F.

The dilaton ϕ and the axion κ parametrize the coset $SL(2,R)/U(1)$. To reveal the $SL(2,R)$ S-duality symmetry in the sector of vector fields (A is inherited from 5D theory, C is the Kaluza-Klein vector) one has to reparametrize them using some dualization [8]. We will reveal S-duality later on the level of the further 3D reduction.

The field equations in terms of the four-dimensional variables read

$$\begin{aligned}
\nabla^2\phi - \mathrm{e}^{2\phi}(\partial\kappa)^2 + \frac{1}{4}\mathrm{e}^{-3\phi}G^2 + \frac{1}{12}\mathrm{e}^{-\phi}\tilde{F}^2 &= 0, &(6)\\
\nabla_\mu\left(\mathrm{e}^{2\phi}\nabla^\mu\kappa\right) - \frac{1}{3}\left[\sqrt{3}\,\nabla_\mu(\mathrm{e}^{-\phi}\tilde{F}^{\mu\nu}C_\nu) + \frac{1}{2}F_{\mu\nu}F^{*\mu\nu}\right] &= 0, &(7)\\
\nabla_\mu\left(\mathrm{e}^{-\phi}\tilde{F}^{\mu\nu} + 2\kappa F^{*\mu\nu}\right) &= 0, &(8)\\
\nabla_\mu\left(\mathrm{e}^{-3\phi}G^{\mu\nu}\right) + \sqrt{3}\,\mathrm{e}^{-\phi}\tilde{F}^{\mu\nu}\partial_\mu\kappa &= 0, &(9)
\end{aligned}$$

and the Bianchi identities are

$$\nabla_\mu F^{*\mu\nu} = 0, \qquad \nabla_\mu G^{*\mu\nu} = 0. \tag{10}$$

It is convenient to introduce the modified Maxwell tensors

$$\begin{aligned}
\mathcal{F}^{\mu\nu} &= \mathrm{e}^{-\phi}\tilde{F}^{\mu\nu} + 2\kappa F^{*\mu\nu}, &(11)\\
\mathcal{G}^{\mu\nu} &= \mathrm{e}^{-3\phi}G^{\mu\nu} + \sqrt{3}\left(\mathrm{e}^{-\phi}\kappa\tilde{F}^{\mu\nu} + \kappa^2 F^{*\mu\nu}\right), &(12)
\end{aligned}$$

in terms of which the Maxwell equations have the divergence form

$$\nabla_\mu\mathcal{F}^{\mu\nu} = 0, \qquad \nabla_\mu\mathcal{G}^{\mu\nu} = 0. \tag{13}$$

B. D=3

Further reduction to $D = 3$ is performed with respect to time, assuming the standard parametrization of the stationary four-metric

$$ds_4^2 = -f(dt - \omega_i dx^i)^2 + f^{-1}h_{ij}dx^i dx^j. \tag{14}$$

The spatial part of the Bianchi identities (10) is solved introducing the electric potentials $C_0 = \bar{v}_1$, $A_0 = \bar{v}_2$, so that

$$G_{i0} = \partial_i\bar{v}_1, \qquad F_{i0} = \partial_i\bar{v}_2. \tag{15}$$

Similarly, the spatial components of the Maxwell equations (13) are solved by introducing magnetic potentials $\bar{u}_1$, $\bar{u}_2$:

$$\mathcal{G}^{ij} = \frac{f}{\sqrt{h}}\,\epsilon^{ijk}\partial_k\bar{u}_1, \qquad \mathcal{F}^{ij} = \frac{f}{\sqrt{h}}\,\epsilon^{ijk}\partial_k\bar{u}_2. \tag{16}$$

The time components of the corresponding equations then give the second order equations for these potentials. Straightforwardly we can find (with the convention $\epsilon^{ijk} = -\epsilon^{0ijk}$)

$$\begin{aligned}
G^{ij} &= \frac{f}{\sqrt{h}}\mathrm{e}^{3\phi}\,\epsilon^{ijk}(w_1)_k, \qquad (w_1)_k := \partial_k\bar{u}_1 - \sqrt{3}\,\kappa(\partial_k\bar{u}_2 - \kappa\partial_k\bar{v}_2), &(17)\\
\tilde{F}^{ij} &= \frac{f}{\sqrt{h}}\mathrm{e}^{\phi}\,\epsilon^{ijk}(w_2)_k, \qquad (w_2)_k := \partial_k\bar{u}_2 - 2\kappa\partial_k\bar{v}_2, &(18)\\
\tilde{F}_{i0} &= (z_2)_i, \qquad (z_2)_i := \partial_i\bar{v}_2 - \sqrt{3}\bar{v}_1\partial_i\kappa. &(19)
\end{aligned}$$

The remaining components of the Maxwell tensors are obtained using the following relations valid for any second rank four-dimensional antisymmetric tensor $W_{\mu\nu}$ with the assumed form of the metric (14):

$$W^{i0} = W^{ij}\omega_j - h^{ij}W_{j0}, \qquad W_{ij} = f^{-2}h_{ik}h_{jl}W^{kl} + 2W_{0[i}\omega_{j]}. \tag{20}$$

Using this we find:

$$
\begin{aligned}
G^{i0} &= \frac{f}{\sqrt{h}}\mathrm{e}^{3\phi}\,\epsilon^{ijk}\omega_j(w_1)_k - \partial^i\bar{v}_1, &(21)\\
G_{ij} &= f^{-1}\sqrt{h}\mathrm{e}^{3\phi}\,\epsilon_{ijk}(w_1)^k + 2\omega_{[i}\partial_{j]}\bar{v}_1, &(22)\\
\tilde{F}^{i0} &= \frac{f}{\sqrt{h}}\mathrm{e}^{\phi}\,\epsilon^{ijk}\omega_j(w_2)_k - (z_2)^i, &(23)\\
\tilde{F}_{ij} &= f^{-1}\sqrt{h}\mathrm{e}^{\phi}\,\epsilon_{ijk}(w_2)^k + 2\omega_{[i}(z_2)_{j]}, &(24)
\end{aligned}
$$

and for the squared quantities:

$$
G^2 = -2(\partial\bar{v}_1)^2 + 2\mathrm{e}^{6\phi}(w_1)^2, \qquad \tilde{F}^2 = -2(z_2)^2 + 2\mathrm{e}^{2\phi}(w_2)^2, \tag{25}
$$

where $(w_1)^2 = (w_1)_i(w_1)^i$.

Now we turn to the Einstein equations:

$$
R_{\mu\nu} - \frac{3}{2}\left(\partial_\mu\phi\partial_\nu\phi + \mathrm{e}^{2\phi}\partial_\mu\kappa\partial_\nu\kappa\right) - \frac{1}{2}\mathrm{e}^{-3\phi}\left(G_{\mu\alpha}G_\nu{}^\alpha - \frac{1}{4}G^2 g_{\mu\nu}\right) - \frac{1}{2}\mathrm{e}^{-\phi}\left(\tilde{F}_{\mu\alpha}\tilde{F}_\nu{}^\alpha - \frac{1}{4}\tilde{F}^2 g_{\mu\nu}\right) = 0. \tag{26}
$$

The Ricci tensor decomposes as follows

$$
\begin{aligned}
R_{00} &= \frac{1}{2}\left(f\nabla^2 f - (\partial f)^2 + \tau^2\right), &(27)\\
R^i{}_0 &= -\frac{f}{2\sqrt{h}}\,\epsilon^{ijk}\partial_j\tau_k, &(28)\\
R^{ij} &= f^2\mathcal{R}^{ij} - \frac{1}{2}\left[\partial^i f\partial^j f + \tau^i\tau^j\right] + h^{ij}R_{00}, &(29)
\end{aligned}
$$

where

$$
\tau^i = -\frac{f^2}{\sqrt{h}}\,\epsilon^{ijk}\partial_j\omega_k. \tag{30}
$$

The $0i$-part of (26) can be solved introducing the twist potential χ via

$$
\tau_i = \partial_i\chi + \frac{1}{2}\left\{\bar{v}_1\partial_i\bar{u}_1 - \bar{u}_1\partial_i\bar{v}_1 + \bar{v}_2\partial_i\bar{u}_2 - \bar{u}_2\partial_i\bar{v}_2 + \sqrt{3}\left[\kappa^2\bar{v}_1\partial_i\bar{v}_2 - \bar{v}_2\partial_i(\kappa^2\bar{v}_1)\right] - \sqrt{3}\left[\kappa\bar{v}_1\partial_i\bar{u}_2 - \bar{u}_2\partial_i(\kappa\bar{v}_1)\right]\right\}. \tag{31}
$$

Using this relation, we can rewrite the 00-component of the Einstein equations as

$$
R_{00} = \frac{1}{4}f\left\{\mathrm{e}^{-3\phi}\left[(\partial\bar{v}_1)^2 + \mathrm{e}^{6\phi}(w_1)^2\right] + \mathrm{e}^{-\phi}\left[(z_2)^2 + \mathrm{e}^{2\phi}(w_2)^2\right]\right\}, \tag{32}
$$

and present the space-space part as

$$
\begin{aligned}
R^{ij} &= \frac{3}{2}f^2 h^{ia}h^{jb}\left(\partial_a\phi\partial_b\phi + \mathrm{e}^{2\phi}\partial_a\kappa\partial_b\kappa\right)\\
&- \frac{1}{2}\mathrm{e}^{-3\phi}f\left[\partial^i\bar{v}_1\partial^j\bar{v}_1 + \mathrm{e}^{6\phi}(w_1)^i(w_1)^j\right] + \frac{1}{4}\mathrm{e}^{-3\phi}fh^{ij}\left[(\partial\bar{v}_1)^2 + \mathrm{e}^{6\phi}(w_1)^2\right]\\
&- \frac{1}{2}\mathrm{e}^{-\phi}f\left[(z_2)^i(z_2)^j + \mathrm{e}^{2\phi}(w_2)^i(w_2)^j\right] + \frac{1}{4}\mathrm{e}^{-\phi}fh_{ij}\left[(z_2)^2 + \mathrm{e}^{2\phi}(w_2)^2\right]. &(33)
\end{aligned}
$$

From the Eq. (28) we then find that the Ricci tensor built on the three-dimensional metric h_{ij} will satisfy the following three-dimensional Einstein equation

$$
\begin{aligned}
\mathcal{R}_{ij} &= \frac{1}{2f^2}\left(\partial_i f\partial_j f + \tau_i\tau_j\right) + \frac{3}{2}\left(\partial_i\phi\partial_j\phi + \mathrm{e}^{2\phi}\partial_i\kappa\partial_j\kappa\right)\\
&- \frac{1}{2f}\left[\mathrm{e}^{-3\phi}\partial_i\bar{v}_1\partial_j\bar{v}_1 + \mathrm{e}^{3\phi}(w_1)_i(w_1)_j + \mathrm{e}^{-\phi}(z_2)_i(z_2)_j + \mathrm{e}^{\phi}(w_2)_i(w_2)_j\right]. &(34)
\end{aligned}
$$