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Classification of integrable super-systems using the SsTools environment

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Classification of integrable super-systems using the SsTools environment ⋆⋆⋆

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Abstract

A classification problem is proposed for supersymmetric evolutionary PDE that satisfy the assumptions of nonlinearity, nondegeneracy, and homogeneity. Four classes of nonlinear coupled boson-fermion systems are discovered under the weighting assumption $|f| = |b| = |D_t| = \frac{1}{2}$. The syntax of the REDUCE package SsTools, which was used for intermediate computations, and the applicability of its procedures to the calculus of super-PDE are described.

Key words: Integrable super-systems, symmetries, recursions, classification, symbolic computation, REDUCE
PACS: 02.30.Ik, 02.70.Wz, 12.60.Jv
1991 MSC: [2000] 35Q53, 37K05, 37K10, 81T40

PROGRAM SUMMARY

Title of program: SsTools

Catalogue number: `xxxx`

Program obtainable from: CPC Program Library, Queen's University of Belfast, N. Ireland; see also [1]

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Licensing provisions: none

Computer for which the program is designed and others on which it has been tested:
Computers: (i) IBM PC, (ii) cluster
Installations: (i) Brock University, St. Catharines, Ontario, Canada L2S 3A1; (ii) SHARCNET `http://www.sharcnet.ca`

Operating system under which the program has been tested: LINUX

Programming language used: REDUCE 3.7, REDUCE 3.8

Memory required to execute with typical data: problem dependent (10 Mb – 1Gb), typical working size < 100 Mb

No. of bits in a word: $32, 64$

No. of processors used: (i) 1; (ii) multiple

Peripherals used: no

Has the code been vectorized? no

No. of lines in distributed program including documentation file, test data, etc.: 2485

Distribution format: ASCII

Online access and tutorials: `http://lie.math.brocku.ca/crack/susy/`

Keywords: Integrable super-systems; symmetries; recursions; classification; symbolic computation; REDUCE

Nature of physical problem: The program allows the classification of $N \geq 1$ supersymmetric nonlinear scaling-invariant evolution equations $\{f_t = \varphi^f$, $b_t = \varphi^b\}$ that admit infinitely many local symmetries propagated by recursion operators; here $b(x, t; \theta)$ is the set of bosonic super-fields and $f(x, t; \theta)$ are fermionic super-fields.

Method of solution: First, (half-)integer weights $|f|$, $|b|$, $\ldots$, $|D_t|$, $|D_x| \equiv 1$ are assigned to all variables and derivatives and then pairs of commuting flows that are homogeneous w.r.t. these weights are constructed. Secondly, the seeds of higher symmetry sequences [2] for the systems are sorted out, and finally the recursion operators that generate the symmetries are obtained [3]. The intermediate algebraic systems upon the undetermined coefficients are solved by using [4].

Restrictions on the complexity of the problem: Computation of symmetries of high differential order for very large evolutionary systems may cause memory restrictions. Additional size/time restrictions may occur if the homogeneity weights of some super-fields are non-positive, see section 1.2 of the Long Write-Up.

Typical running time: depends on the size and complexity of the input system and varies between seconds and minutes.

Unusual features of the program: SsTools has been extensively tested using hundreds of PDE systems within three years on UNIX-based PC-machines. SsTools is applicable to the computation of symmetries, conservation laws, and Hamiltonian structures for $N \geq 1$ evolutionary super-systems with any N. SsTools is also useful for performing extensive arithmetic of general nature including differentiations of super-field expressions.

References:
[1] `http://lie.math.brocku.ca/crack/susy/sstools.red`; The support package Crack is obtained from `http://lie.math.brocku.ca/crack/src/crack.tar.gz`
[2] P.J. Olver, Applications of Lie groups to differential equations, 2nd ed., Springer, Berlin, 1993.
[3] I. S. Krasil'shchik, P. H. M. Kersten, Symmetries and recursion operators for classical and supersymmetric differential equations, Kluwer, Dordrecht, 2000.
[4] T. Wolf, Applications of Crack in the classification of integrable systems, CRM Proc. Lecture Notes 37 (2004), 283–300.

LONG WRITE–UP

Introduction

The principle of symmetry belongs to the foundations of modern mathematical physics. The differential equations that constitute *integrable* [1] models practically always admit symmetry transformations and, reciprocally, new classes of integrable phenomena in physics are obtained by postulating some symmetry invariance. The presence of symmetry transformations in a system yields two types of explicit solutions: those which are invariant under a transformation (sub)group and, secondly, the solutions obtained by propagating a known solution by the same group. The two schemes for constructing new solutions of PDE are crucial for systems of super-equations of mathematical physics (e.g., supergravity models); these equations involve commuting (bosonic, or 'even') and anticommuting (fermionic, or 'odd') independent variables and/or unknown functions. Indeed, other methods for solving nonlinear equations need a special adaptation to the super-field setting, see e.g. [2]; another approach to integrability of supersymmetric equations has been investigated in [3]. Hence the symmetry considerations [1], which are based on the computation of infinitesimal symmetry generators for PDE, become highly important. The arising computational problems are unmanageable without computer algebra that permits handling relevant systems.

In the literature it has been observed (see [1,4] and references therein) that the principal phenomena in nature are governed by systems of PDE that admit higher symmetries, that is, the symmetry transformations that involve higher order derivatives of the unknown functions. Simultaneously, when dealing with supersymmetric models of theoretical physics, it is often hard to predict whether a certain mathematical approximation will be truly integrable or not. Therefore we apply the symbolic computational approach to the physical problem of classifying the systems that exhibit necessary integrability features. It must be noted that the very idea to filter out integrable cases using the presence of 'many' symmetries is widely accepted in the computer branch of modern mathematical physics, see e.g. [4,5]. These systems are called symmetry integrable [1,4]); in some cases, they can be transformed to exactly solvable equations or their extensions. In view of this classification task, we analyzed fermionic extensions of the Burgers and Boussinesq equations and related the former with evolutionary systems on associative algebras in [6].

This paper is organized as follows. First we formulate the axioms of the classification problem for $N = 1$ supersymmetric systems of evolutionary PDE. In sections 1.1 and 1.2 we describe the two modes of the procedure `ssym` in the package SsTools that allow, respectively, finding unordered pairs of commuting flows and the computation of symmetries for previously found systems. Also, in section 1.2 we review a geometric (coordinate-free) algorithm for constructing recursion operators. However, we refer to [5,7] for basic notions and concepts in the geometry of (super)PDE, see also [8–10,16] and references therein. The principal result of this paper is that there exist only four nonlinear coupled boson-fermion systems that satisfy the axioms and the weighting $|f| = |b| = |D_t| = \frac{1}{2}$. Section 1 concludes with the classification of their recursion operators. In the following sections we investigate properties of these four systems, giving simultaneously the syntax and describing applicability of SsTools subroutines developed for the calculation of the scaling weights, symmetries, linearizations, conservation laws needed for introducing the nonlocalities, and recursion operators. Finally, sample runs of SsTools are given.

1 The classification problem

Let us introduce some notation. We denote by θ the super-variable and we put $\mathcal{D} \equiv D_\theta + \theta\, D_x$ such that $\mathcal{D}^2 = D_x$ and $[\mathcal{D}, \mathcal{D}] = 2D_x$; here D_θ and D_x are the