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Railway Track Allocation by Rapid Branching

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Abstract

The *track allocation problem*, also known as train routing problem or train timetabling problem, is to find a conflict-free set of train routes of maximum value in a railway network. Although it can be modeled as a standard path packing problem, instances of sizes relevant for real-world railway applications could not be solved up to now. We propose a rapid branching column generation approach that integrates the solution of the LP relaxation of a path coupling formulation of the problem with a special rounding heuristic. The approach is based on and exploits special properties of the bundle method for the approximate solution of convex piecewise linear functions. Computational results for difficult instances of the benchmark library TTPLIB are reported.

1 Introduction

Routing a maximum number of trains in a conflict-free way through a track network is one of the basic scheduling problems for a railway company. This *optimal track allocation problem*, also known as train routing problem or train timetabling problem, has received growing attention in the operations research literature, see [8, 2, 11, 6, 17] for some recent references. A branch on the study of advanced models that incorporate, e.g., additional robustness aspects, has already been started, see, e.g., [12]. However, the problem remains that up to now the basic problem can hardly be solved even for small instances. Corridors or single stations mark or are quickly beyond the limits of the current solution technology, such that network optimization problems can not be addressed.

Finding a good track allocation model is a key prerequisite for progress towards the solution of large-scale track allocation problems. The authors of [4] proposed a novel path coupling formulation based on train path and track configuration variables. The model provides a strong LP bound, is amenable to standard column generation techniques, and therefore suited for large-scale

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computation. Indeed, it was shown that LP relaxations of large-scale track allocation problems involving hundreds of potential trains could be solved to proven or near optimality in this way. However, similar results for integer solutions could not be provided at that time.

This topic is addressed in this paper. Extending the work in [4], we present a sophisticated solution approach that is able to compute high-quality integer solutions for large-scale railway track allocation problems. Our algorithm is an adaptation of the rapid branching method introduced in [3] (see also the thesis [20]) for integrated vehicle and duty scheduling in public transport. The method solves a Lagrangean relaxation of the track allocation problem as a basis for a branch-and-generate procedure that is guided by approximate LP solutions computed by the bundle method. This successful second application provides evidence that rapid branching is a general solution method for large-scale path packing and covering problems.

The paper is structured as follows. Section 2 recapitulates the track allocation problem and the path configuration model. Section 3 discusses the solution of an associated Lagrangean relaxation by the bundle method. In Section 4 we adapt the rapid branching heuristic to deal with track allocation (maximization) problems. Section 5 reports computational results. We demonstrate that rapid branching can be used to produce high quality solutions for large-scale track allocation problems.

2 The Track Allocation Problem

We briefly recall in this section a formal description of the *track allocation problem*; more details can be found in the articles [5, 8, 2]. Consider an acyclic digraph $D = (V, A)$ that represents a time-expanded railway network. Its nodes represent arrival and departure events of trains at a set S of stations at discrete times $T \subseteq \mathbb{Z}$, its arcs model activities of running a train over a track, parking, or turning around. Let I be a set of requests to route trains through D. More precisely, train $i \in I$ can be routed on a path through some suitably defined subdigraph $D_i = (V_i, A_i) \subseteq D$ from a starting point $s_i \in V_i$ to a terminal point $t_i \in V_i$. Denote by P_i the set of all routes for train $i \in I$, and by $P = \bigcup_{i \in I} P_i$ the set of all train routes (taking the disjoint union).

Let $s(v) \in S$ be the station associated with departure or arrival event $v \in V$, $t(v)$ the time, and $J = \{s(u)s(v) : (u, v) \in A\}$ the set of all railway tracks. An arc $(u, v) \in A$ *blocks* the underlying track $s(u)s(v)$ for the time interval $[t(u), t(v)[$, and two arcs $a, b \in A$ are *in conflict* if their respective blocking time intervals overlap. Two train routes $p, q \in P$ are in conflict if any of their arcs are in conflict. A *track allocation* or timetable is a set of conflict-free train routes, at most one for each request set. Given arc weights w_a, $a \in A$, the weight of route $p \in P$ is $w_p = \sum_{a \in p} w_a$, and the weight of a track allocation $X \subseteq P$ is $w(X) = \sum_{p \in X} w_p$. The *track allocation problem* is to find a conflict-free track allocation of maximum weight.

The track allocation problem can be modeled as a multi-commodity flow problem with additional packing constraints, see [8, 2, 11]. This model is computationally difficult. We consider in this article an alternative formulation as a *path coupling problem* based on 'track configurations' as proposed by the authors of [4]. A valid configuration is a set of arcs on some track $j \in J$ that are mutually not in conflict. Denote by Q_j the set of configurations for track $j \in J$, and by $Q = \bigcup_{j \in J} Q_j$ the set of all configurations. Introducing 0/1-variables x_p, $p \in P$, and y_q, $q \in Q$, for train paths and track configurations, the track allocation problem can be stated as the following integer program:

$$
\begin{array}{llrlrr}
\text{(PCP)} & \max & \sum\limits_{p \in P} w_p x_p & & & \text{(i)} \\
& \text{s.t.} & \sum\limits_{p \in P_i} x_p & \leq 1, & \forall\, i \in I & \text{(ii)} \\
& & \sum\limits_{q \in Q_j} y_q & \leq 1, & \forall\, j \in J & \text{(iii)} \\
& & \sum\limits_{a \in p \in P} x_p - \sum\limits_{a \in q \in Q} y_q & \leq 0, & \forall\, a \in A & \text{(iv)} \\
& & x_p, y_q & \geq 0, & \forall\, p \in P,\ q \in Q & \text{(v)} \\
& & x_p, y_q & \in \{0,1\}, & \forall\, p \in P,\ q \in Q. & \text{(vi)}
\end{array}
$$

The objective PCP (i) maximizes the weight of the track allocation. Constraints (ii) state that a train can run on at most one route, constraints (iii) allow at most one configuration for each track. Inequalities (iv) link train routes and track configurations to guarantee a conflict-free allocation, (v) and (vi) are the non-negativity and integrality constraints. Note that the upper bounds $x_p \leq 1$, $p \in P$, and $y_q \leq 1$, $q \in Q$, are redundant.

Introducing appropriately defined matrices $A \in \mathbb{Q}^{I \times P}$, $B \in \mathbb{Q}^{J \times Q}$, $C \in \mathbb{Q}^{I \times A}$, $D \in \mathbb{Q}^{J \times A}$, and a weight vector $w \in \mathbb{Q}^P$, program (PCP) can be stated in matrix form as follows:

$$
\text{(PCP)} \ \max\ w^T x,\ Ax = \mathbf{1},\ By = \mathbf{1},\ Cx - Dy \leq 0,\ (x, y) \in \{0,1\}^{P \times Q}.
$$

The authors of [4] have shown that train path and track configuration variables can be priced by solving shortest path problems in suitably defined acyclic digraphs, such that the LP relaxation of program (PCP) can be solved in polynomial time.

3 A Bundle Approach

The PCP consists of a train routing and a track configuration sub-model that are linked by coupling constraints. The sub-models are easy, but time consuming to solve using a column generation procedure based on acyclic shortest path computations, the coupling constraints are simple but numerous. This combinatorial structure can be exploited using a Lagrangean relaxation approach in which, of course, precision and speed of convergence are critical issues. It turns out that the bundle method fits perfectly with such a scheme.

A Lagrangean dual of model PCP arises from a Lagrangean relaxation of the coupling constraints PCP (iv) and a relaxation of the integrality constraints PCP (vi) and (vii):

$$\text{(LD)} \quad \min_{\lambda \geq \mathbf{0}} \left[\max_{\substack{Ax=\mathbf{1},\\ x\in[0,1]^P}} (w^\mathsf{T} - \lambda^\mathsf{T} C)x + \max_{\substack{By=\mathbf{1},\\ y\in[0,1]^Q}} (\lambda^\mathsf{T} D)y \right].$$

LD is equivalent to the dual of the LP relaxation of PCP, and hence provides upper bounds for PCP. Introducing functions

$$\begin{aligned} f_P &: \mathbb{R}^A \to \mathbb{R}, \quad \lambda \mapsto \max(w^\mathsf{T} - \lambda^\mathsf{T} C)x,\ Ax = \mathbf{1},\ x \in [0,1]^P \\ f_Q &: \mathbb{R}^A \to \mathbb{R}, \quad \lambda \mapsto \max(\lambda^\mathsf{T} D)y,\ By = \mathbf{1},\ y \in [0,1]^Q \\ f_{P,Q} &:= f_P + f_Q, \end{aligned}$$

LD can be stated more shortly as follows:

$$\text{(LD)} \quad \min_{\lambda \geq \mathbf{0}} f_{P,Q}(\lambda) = \min_{\lambda \geq \mathbf{0}} \left[f_P(\lambda) + f_Q(\lambda) \right].$$

The functions f_P and f_Q are convex and piecewise linear. Their sum $f_{P,Q}$ is therefore a decomposable, convex, and piecewise linear function; $f_{P,Q}$ is, in particular, non-smooth. This is precisely the setting for an application of the *proximal bundle method* (PBM) to a maximization problem, see [14, 15, 13, 3, 20] for details.

When applied to LD, the PBM constructs cutting plane models of the functions f_P and f_Q in terms of subgradient bundles J_P^i and J_Q^i that are used to produce two sequences of iterates $\lambda^i, \mu^i \in \mathbb{R}^A$, $i = 0, 1, \ldots$. The points μ^i are called *stability centers*; they converge to a solution of LD. The points λ^i are trial points calculated by solving a quadratic program over a trust region around the current stability center, whose size is controlled by some positive weight u:

$$(QP_{P,Q}^i) \qquad \lambda^{i+1} := \operatorname*{argmin}_{\lambda} f_{P,Q}(\lambda) - \tfrac{u}{2} \left\| \mu^i - \lambda \right\|^2. \tag{1}$$

A function evaluation at a trial points results either in a shift of the stability center, or in an improvement of the cutting plane model. A key point is that the high-dimensional quadratic program $(QP_{P,Q}^i)$ (the dimension is equal to the number of coupling constraints) has a dual whose dimension coincides with the number subgradients in the current bundle. The method converges for a bundle size of two, typical sizes in practice are around 10 or 15. This dimension reduction makes the problem computationally tractable.