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A Visual Approach to Analysis of Stress Tensor Fields

Submitted August 31, 2010

ZIB-Report 10-26 (August 2010)

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Abstract—We present a visual approach for the exploration of stress tensor fields. Therefore, we introduce the idea of multiple linked views to tensor visualization. In contrast to common tensor visualization methods that only provide a single view to the tensor field, we pursue the idea of providing various perspectives onto the data in attribute and object space. Especially in the context of stress tensors, advanced tensor visualization methods have a young tradition. Thus, we propose a combination of visualization techniques domain experts are used to with statistical views of tensor attributes. The application of this concept to tensor fields was achieved by extending the notion of shape space. It provides an intuitive way of finding tensor invariants that represent relevant physical properties. Using brushing techniques, the user can select features in attribute space, which are mapped to displayable entities in a three-dimensional hybrid visualization in object space. Volume rendering serves as context, while glyphs encode the whole tensor information in focus regions. Tensorlines can be included to emphasize directionally coherent features in the tensor field. We show that the benefit of such a multi-perspective approach is manifold. Foremost, it provides easy access to the complexity of tensor data. Moreover, including well-known analysis tools, such as Mohr diagrams, users can familiarize themselves gradually with novel visualization methods. Finally, by employing a focus-driven hybrid rendering, we significantly reduce clutter, which was a major problem of other three-dimensional tensor visualization methods.

Index Terms—

1 Introduction

The focus of this work is the analysis and visualization of 3D stress tensor fields, which express the response of a material to applied forces. Important application areas and their interest in such data are: In material science, a material's behavior under pressure is observed to examine its stability. Similar questions also arise in astrophysics. Rock fractures caused by tension or compression, for example, are analyzed in geosciences. A medical example is the simulation of an implant design's impact on the distribution of physiological stress inside a bone [1]. Common to most of these areas is the goal of finding regions where the inspected material tends to crack. Various failure models exist, but in general they are based on the analysis of large shear stresses. Besides understanding a physical phenomenon, tensor analysis can help to detect failures in simulations where tensors appear as intermediate product. In all these application areas, regions of interest are not necessarily known in advance. For this reason, powerful visual exploration and analysis tools are of high importance.

The complexity of tensor data makes them hard to visualize and interpret. Therefore, users tend to analyze tensor data via two-dimensional plots of derived scalars (data reduction). Although these plots simplify the analysis at first glance, they do not communicate the evolution of tensors over the whole field [2]. They might even fail to convey all information given by a single tensor. From a visualization point of view, the difficulty lies in depicting each tensor's complex information, especially for three-dimensional tensor fields. Often, visualizations are restricted to two-dimensional slices (data projection), as three-dimensional visualizations tend to result in cluttered images. However, data reduction and data projection both reduce the complex information of the tensor field to a small subset. Thus, the richness of the data is not communicated.

A further challenge, for example in contrast to vector field visualization, is the young tradition of advanced tensor visualization methods in the considered application areas. Users need to get used to the advantages of modern visualization techniques, and therefore need tools to *explore* the data so they can develop an intuition and construct new hypotheses. Therefore, it is important to link methods domain experts are already used to with novel techniques. The main challenges in visualizing three-dimensional tensor fields, and the resulting goals of our work are:

- Tensor data are hard to interpret. Thus, we provide an intuitive approach to the analysis of tensor data.
- Tensor visualization methods do not have a long tradition in their respective application areas. Thus, we provide well-known perspectives onto these data and link them with novel visualization methods.
- A lack of a-priori feature definitions prevents the use of automatic segmentation algorithms. Thus, we allow users to *find the unknown* and let them steer the visualization process.
- The stress tensors we are dealing with are symmetric 3D tensors described by six independent variables. Thus, effectively capturing all of this information with a single visualization method is practically not feasible. We therefore employ a feature-dependent hybrid visualization.

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Contribution

To meet these goals, we present a new access to tensor fields. The major contributions of this paper are:

- *Introduction of shape space theory as basic means for feature designation in attribute space.* Previous work mostly concentrated on the properties of a specific tensor type. We introduce an intuitive way of finding tensor invariants that reflect relevant features. Building upon the idea of shape space, the challenging task of translating questions into appropriate invariants boils down to a basis change of shape space. Using concepts from stress analysis and including failure models, we present invariants for stress tensor fields together with common and new visualization techniques (Figure 2). However, our approach is extendable to the analysis of various types of symmetric second-order tensors.

- *Introduction of multiple linked views to tensor visualization.* Previous work mostly concentrated on only two dimensions and/or one particular visualization technique. We pursue the idea of providing various perspectives onto the data and propose visual exploration in attribute and object space. The concept of shape space serves as link between the abstract tensor and its visualization in attribute space. In object space, features are mapped to displayable entities and are explored in a three-dimensional hybrid visualization.

2 Related Work

Besides work from tensor field visualization [3], our work is based on publications from multiple view systems [4] as well as from the visualization of multivariate data [5]. This review is structured according to our main contributions focusing on second-order stress tensors and their visualization in attribute (diagram views) and object space (spatial views).

Tensor Invariants: Central to our work is the finding that tensor visualization methods can be designed and parametrized by a specific choice of invariants, which are scalar quantities that do not change under orthogonal coordinate transformation. Considering and analyzing important invariants is common in many physical applications [6]. For analysis of diffusion tensors, [6] has been transferred to visualization [7]. In the same context, Bahn [8] came up with the definition of *eigenvalue space*, where the eigenvalues are considered to be coordinates of a point in Euclidean space. In this work, we use the term *shape space* referring to application areas such as vision and geometric modeling. Coordinates within this space describe a set of tensor invariants and are called *shape descriptor*.

Diagram Views: Only few visualization papers are related to using diagram views for tensors. Mohr's circle [9] is a common tool in material mechanics, being used to compute coordinate transformations. In visualization, it has been applied to diffusion tensors to depict the tensor's diffusivity [10] as well as to stress tensors [11]. Being a known technique for domain experts, Mohr diagrams can ease the access to novel visualization methods. Directional histograms have been used to visualize the distribution of fiber orientations in sprayed concrete [12] and for diffusion tensors in terms of rose diagrams and 3D scatterplots of the major eigenvector angles [13]. To the best of our knowledge, combined views for tensors have not been presented previously.

Spatial Views: A common classification of spatial visualization methods for second-order tensors is to distinguish between local, global and feature-based methods.

Local methods use geometries (glyphs) to depict single tensors at discrete points. Shape, size, color and transparency are used to encode tensor invariants. Dense glyph visualizations use less complex geometries together with placement algorithms [14], [15], [16]. When only selected locations are examined (probing), more complex geometries can be used. A variety of glyph types have been presented, focusing on stress tensors [2], higher-order tensors [17] and perceptual issues [18], [19]. Although, local methods have the potential to depict the whole tensor information, they generally fail in giving an overview of the complete 3D tensor field.

In contrast, global methods present an overview and emphasize regional coherence. They can be classified into methods based on scalar and vector visualization, as well as hybrid methods. Scalar visualization methods that are used to visualize tensors are ray-casting [20], [21], [22] and splatting [23], [24]. The main challenge is the design of an appropriate transfer function. Kindlmann et al. [20], [21] define an opacity transfer function based on the isotropic behavior of the tensor field. Color and shading are defined by tensor properties such as orientation and shape. Inspired by this work, Hlawitschka et al. [22] focus on directional information for transfer function design to emphasize fiber bundle boundaries. Recently, Dick et al. [1] presented a colormapping for stress tensors in order to distinguish between compressive and tensile forces.

Vector visualization methods are used to depict the behavior of the eigenvectors. We distinguish line tracing algorithms like tensorlines [25], texture-based approaches such as Line Integral Convolution (LIC) [26] and reaction-diffusion textures [27], [21]. Hotz et al. [28] presented a LIC-like method for the visualization of two-dimensional slices of a stress tensor field. They introduce a mapping of the indefinite stress tensor to a positive-definite metric. The mapped eigenvalues then are used to define input parameters used for LIC.

Whereas scalar-related visualization techniques are able to cover aspects of the whole 3D field, vector-related methods are mostly restricted to two dimensions. Hybrid approaches combine global and local methods [29], [30] as well as scalar- and vector-related techniques [1], [31]. Dick et al. [1] proposed hybrid visualization for 3D stress tensor fields. They combine ray-casting of the three eigenvalues with tensorlines to depict selected directions. To account for clutter, tensorlines are only seeded on a surface mesh. Although some hybrid approaches try to combine complex focus with non-disruptive context visualization, none of the existing methods allows the analysis and visualization of a complete 3D field both in detail and at

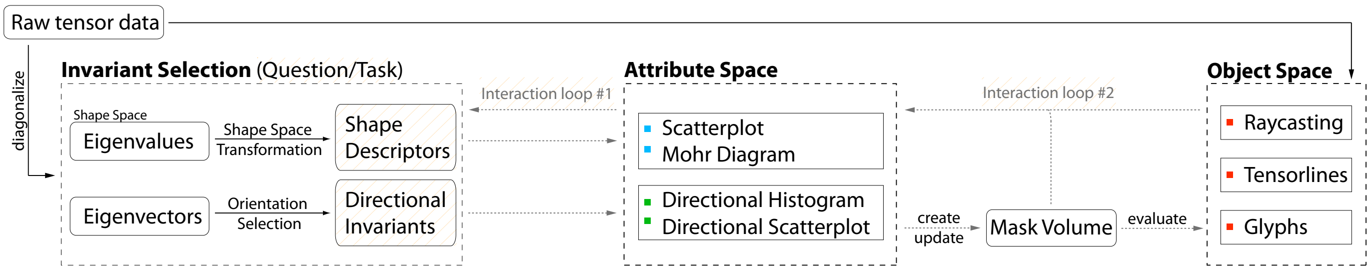

Fig. 1: *Tensor analysis and visualization pipeline. The basis builds the diagonalization of the tensor into its eigenvalues and eigenvectors. The first step then is the choice of appropriate shape descriptors and directional invariants steered by a specific question or task. These are visualized in attribute space. Within this space, features are selected using brushing techniques and are then encoded in a mask volume. In object space, various tensor visualization techniques are combined in a feature-driven hybrid visualization. Volume rendering provides context, and glyphs or tensorlines are placed in focus regions. The user has a variety of options to adjust the visualization (interaction loops): Shape descriptors and directional invariants can be adapted (#1) and focus/context regions can be interactively refined (#2).*

large.

Feature-based methods comprise topological methods [32], [33], [34], [35] and tensor segmentation algorithms [36], [37]. Regions of similar behavior are merged, which helps to handle the complex information within a tensor field. However, automatic segmentation algorithms can only be used, if the characteristics of interesting structures can be defined in advance. They fail in describing new features and might even remove important aspects of the data [5]. Furthermore, they are hard to extend to three-dimensional tensor data.

3 Tensor Visualization and Analysis Pipeline

Multiple linked views are used to explore three-dimensional stress tensor fields. We distinguish between diagram views in *attribute space* (see Section 5) and three-dimensional spatial views in *object space* (Section 6). Both are linked over a *mask volume*, i.e., a three-dimensional data structure of the same size as the input data storing a binary value (0 or 1). The mask is created and modified by brushing tensor properties in attribute space; it is evaluated for rendering in the spatial domain. For an overview of the proposed pipeline see Figure 1.

The basis of the pipeline is the diagonalization of the tensor (Section 4.1.1). Thus, the tensor is decomposed into *shape* and *orientation*, whereas shape refers to the eigenvalues and orientation to the eigenvectors. The first step then is the choice of appropriate shape descriptors and directional invariants (Section 4). We conceive this process as translating a question into a mathematical description (Figure 2). Being supported by various views in attribute space, the user can select and substitute tensor properties (interaction loop #1) until a set is found for being explored in more detail. Multiple views are possible at the same time, so different parameter choices and selections can be visually compared. Within these views, features are selected and highlighted using brushing-and-linking techniques (interaction loop #2). In this work, we propose the following diagrams: Shape space scatterplots can be understood as a cut through three-dimensional shape space and, thus, deliver insight into the distribution of tensor properties (Section 5.1). Mohr diagrams [11] represent the most important invariants for stress tensors (Section 4.4). Directional histograms are used to analyze the distribution of principal directions (Section 5.3), and directional scatterplots to inspect shape properties together with directions (Section 5.4).

Hybrid object space rendering (Section 6) allows the inspection of the selected features in a spatial context. The mask defines in which regions glyphs are displayed and/or tensorlines are started. Volume rendering of scalar invariants serves as a context view. If the final image does not show all relevant features, users may refine their selections in attribute space, changing the mask volume and the rendering accordingly. Selections in object space, for example of single glyphs, are part of our future work.

4 Tensor Invariants and Shape Space

In this section we formulate the task of finding relevant features in the language of shape space. Then we discuss our particular choice of shape descriptors, directions (Section 4.4) and shape space scaling (Section 4.3) for stress tensor fields.

4.1 Foundations

For the three-dimensional Euclidean space, a tensor T with respect to a basis (b_1, b_2, b_3), denoted by T_b, can be described by a matrix $M \in \mathbb{R}^{3\times 3}$. That is, $T_b = M = (m_{ij})$ with i,j = 1,2,3. A tensor field over some domain D assigns a tensor $T(x)$ to every point $x \in D$.

Tensor invariants are scalar quantities that do not change under orthogonal coordinate transformation. In general, any scalar function $f(\lambda_1, \lambda_2, \lambda_3)$ again is an invariant. Most common examples are the tensor's eigenvalues, determinant and trace.

Question	Invariants		Scaling	Visualization Technique	Space	
	Shape	Orientation			Attribute	Object
Distinguish regions of compression, expansion, shear and isotropic regions.	$\sigma_1, \sigma_2, \sigma_3$		Logarithmic (SP)	Scatterplot	X	
Explore regions of high shear and kind of anisotropy.	τ, R		Logarithmic (SP)	Scatterplot	X	
Distinguish regions of compression, expansion, shear. Kind of anisotropy. Tensor as a whole.	τ, c, R		Linear (SP)	Mohr Diagram	X	
Distribution of principal directions.		e_1, e_2, e_3	-	Directional Histogram	X	
Distribution of directions of maximum shear stress.		$(e_1 \pm e_3)$	-	Directional Histogram	X	
Provide a spatial context by means of a derived scalar field.	$f(\sigma_1, \sigma_2, \sigma_3)$		Linear (SP)	Volume Rendering		X
Encode whole tensor information in focus regions.	$\sigma_1, \sigma_2, \sigma_3$	e_1, e_2, e_3	Asymmetric (POS)	Ellipsoid Glyph		X
Depict the normal stress acting at a given point in any direction.	$\sigma_1, \sigma_2, \sigma_3$	e_1, e_2, e_3	Linear (SP)	Reynolds Glyph		X
Depict the magnitude of shear stress acting at a given point in any direction.	τ	$(e_1 \pm e_3)$	Linear (SP)	HWY Glyph		X
Emphasize selected directions.		e_1, e_2, e_3	-	Tensorlines		X
DTI example: Which regions exhibit high anisotropy, and do they have a characteristic shape?	FA, mode(T)	e_1, e_2, e_3	Normalized	Superquadric Glyph		X

Fig. 2: *Invariant Selection. The table gives examples for shape descriptors and directional invariants that correspond to a specific task or question. We mainly present invariants for stress tensors. However, our approach is extendable to various types of symmetric second-order tensors. Besides convertible invariants, the analysis of tensors from diverse application areas requires variable scalings. The abbreviations SP and POS refer to sign-preserving mappings (SP) and mappings into a positive-definite metric (POS), respectively. Furthermore, the table lists possible visualization techniques in attribute and object space.*

4.1.1 Tensor Diagonalization

Tensors are *invariant* under coordinate transformation, which distinguishes them from matrices. That is, the characteristics of the tensor stay the same, independent from the choice of basis. Consequently, a tensor can be analyzed using any convenient coordinate system.

In the following, we only consider symmetric tensors, i.e., $m_{ij} = m_{ji}$, being defined by six independent components. They can be transformed into a principal coordinate system using the concept of eigenanalysis

$$U\;T\;U^T = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{1}$$

The diagonal elements λ_i are the eigenvalues and the transformation matrix U is composed of the eigenvectors e_i. For symmetric tensors, the eigenvalues are all real, and the eigenvectors constitute an orthonormal basis. They are ordered such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

4.1.2 Stress Tensor

A stress tensor conveys information about the stress acting on cutting planes through a material (Figure 3). It is given as

$$\sigma = \begin{pmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{pmatrix}, \tag{2}$$

with the diagonal components σ_{ij} being the *normal stress* components and the off-diagonal components τ_{ij} the *shear stress* components respective to cutting planes normal to the coordinate axis. The sign of the normal stress components

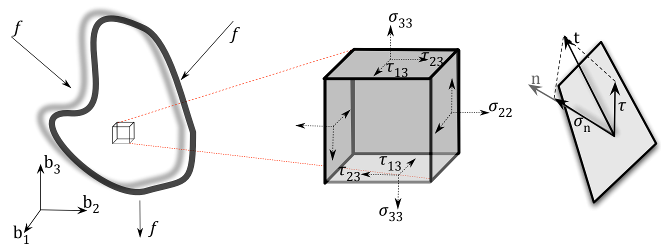

Fig. 3: *External forces f that are applied to a material (left), stress measured on an infinitesimally small volume element (middle), and force (traction t) acting on an cutting plane with normal vector n (right).*

encodes if they are compressive or tensile. In this paper, we interpret negative eigenvalues as compressive forces (making the volume smaller) and positive eigenvalues as tensile forces (expanding the volume). It is worth noting that in some application areas the sign is interpreted in a reverse way. If forces are balanced and there is no rotation (which is, in general, fulfilled for infinitesimally small volume elements), the tensor is *symmetric* and uniquely described by its three eigenvalues and eigenvectors (Equation (1)). In this context, the eigenvectors are called principal stress axes, and the eigenvalues are called principal stresses. As principal stresses may be positive or negative, the tensor is indefinite. The force (traction vector) t acting on a cutting plane with normal vector n is given by

$$t = \sigma \cdot n = \tau + \sigma_n. \tag{3}$$

It can be decomposed into its normal stress σ_n and shear stress component τ (Figure 3, right). In cutting planes orthogonal