Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

Michael Poss[1]
Christian Raack[2]

# Affine recourse for the robust network design problem: between static and dynamic routing[3]

[1]Department of Computer Science, Faculté des Sciences, Université Libre de Bruxelles, Brussels, Belgium, mposs@ulb.ac.be
[2]Zuse Institute Berlin (ZIB), Takustr. 7, D-14195 Berlin, Germany, raack@zib.de
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# Affine recourse for the robust network design problem: between static and dynamic routing

Michael Poss [§]     Christian Raack [¶]

**Abstract**

Affinely-Adjustable Robust Counterparts provide tractable alternatives to (two-stage) robust programs with arbitrary recourse. We apply them to robust network design with polyhedral demand uncertainty, introducing the affine routing principle. We compare the affine routing to the well-studied static and dynamic routing schemes for robust network design. All three schemes are embedded into the general framework of two-stage network design with recourse. It is shown that affine routing can be seen as a generalization of the widely used static routing still being tractable and providing cheaper solutions. We investigate properties on the demand polytope under which affine routings reduce to static routings and also develop conditions on the uncertainty set leading to dynamic routings being affine. We show however that affine routings suffer from the drawback that (even totally) dominated demand vectors are not necessarily supported by affine solutions. Uncertainty sets have to be designed accordingly. Finally, we present computational results on networks from SNDlib. We conclude that for these instances the optimal solutions based on affine routings tend to be as cheap as optimal network designs for dynamic routings. In this respect the affine routing principle can be used to approximate the cost for two-stage solutions with free recourse which are hard to compute.

**Keywords:** Robust optimization; Network design; Recourse; Affine Adjustable Robust Counterparts; Affine Routing; Demand polytope.

[§]Department of Computer Science, Faculté des Sciences, Université Libre de Bruxelles, Brussels, Belgium, mposs@ulb.ac.be

[¶]Zuse Institute Berlin (ZIB), Takustr. 7, D-14195 Berlin, Germany, raack@zib.de. This research has been supported by the DFG research Center Matheon

# 1 Introduction

In the classical deterministic network design problem, a set of point-to point commodities with known demand values is given, and capacities have to be installed on the network links at minimum cost such that the resulting capacitated network is able to accommodate all demands simultaneously by a multi-commodity flow. When demands are known with precision, this problem has been studied extensively in the literature, involving decompositions algorithms, extended formulations and strong cutting planes, see [5, 18, 22, 23, 25, 41], among others.

In practice however, exact demand values are usually not known at the time the design decisions must be made. In telecommunications, demands are estimated which can be done for instance by using traffic measurements (see [45], among others) or population statistics, see [19, 24]. These estimations allow the problem to be formulated and solved using classical tools of deterministic mathematical programming. However, the actual traffic forecast is strongly overestimated in order to yield robust networks capable of routing potential traffic peaks. Such overestimation results in overprovisioned networks wasting capital as well as operational expenditures such as energy resources.

Robust optimization overcomes this problem by explicitly taking into account the uncertainty of the data already in the modeling introducing so-called *uncertainty sets*. A solution is said to be feasible if it is feasible for all realizations of the data in a predetermined uncertainty set $\mathcal{D}$, see [10, 11, 17, 44]. The original framework of Soyster [44] assumes that all decisions must be identical for all values of the uncertain parameters. Introducing more flexibility, two-stage robust optimization allows to adjust a subset of the problem variables only after observing the actual realization of the data, see [12]. In fact, it is natural to apply this two-stage approach to network design since very often first stage capacity design decisions are made in the long term while the actual routing is adjusted based on observed user demands. This second stage adjusting procedure is called *recourse* which in the context of (telecommunications) network design relates to what is known as traffic engineering. Unrestricted second stage recourse in robust network design is called *dynamic routing*, see [20, 28], and [34]. Given a fixed design, the commodity routing can be changed arbitrarily for every realization of the demands. Chekuri et al. [20] and Gupta et al. [28] show that allowing for dynamic routing makes robust network design intractable. Already deciding whether or not a fixed capacity design allows for a dynamic routing of demands in a given polytope is $\mathcal{NP}$-complete (on directed graphs).

Even more general, Ben-Tal et al. [12] observe that two-stage robust linear programming with free recourse is computationally intractable and suggest to limit the flexibility in the second stage by affine functions which makes the problem tractable. Chen and Zhang [21] extend this idea by using extended formulations of uncertainty sets and by applying affine recourse in the resulting higher-dimensional variable spaces.

Interestingly, this limitation in the flexibility of the second stage recourse has been used earlier in robust network design without relating it to two-stage optimization. Ben-Ameur and Kerivin [8, 9] introduce the concept of *static routing* (or oblivious routing).After fixing the design, the routing of a commodity is allowed to change but the actual flow has to be a linear function of the observed demand of the same commodity. Static routing results in a fixed set of paths for every commodity and also a fixed percental splitting of flow among these paths independent of the realization of the commodity demand. In this context we speak of a *routing template* used by all demand realizations. Robust network design with static routing can be handled as a single-stage problem introducing template variables, see the next section. Static routing has been used extensively since then. Ordóñez and Zhao [37] study structural properties of the robust network design problem when cost and demand values belong to conic uncertainty sets. Altin et al. [1] develop a compact integer linear programming model for virtual private network design with continuous capacities and single path routing using. Altin et al. [2, 3] and Koster et al. [32] study the network design problem assuming splittable flow and integer capacities. Polyhedral investigations and computational evaluations of the models are carried out.

The restriction of routing templates for every commodity makes the problem tractable but it is of course very conservative in terms of capacity cost compared to dynamic routing. Recently there have been several attempts to handle less restrictive routing principles. These could be shown to be intractable just like the dynamic case. Ben-Ameur [7] partitions the demand uncertainty set into two (or more) subsets using hyperplanes and devises specific routings for each subset. The resulting optimization problem is $\mathcal{NP}$-hard when no assumptions is made on the hyperplanes. Ben-Ameur thus discusses simplifications where either the entire hyperplane or its direction is given. Scutellà [42] allows for two (or more) routing templates to be used conjointly. These routing templates are devised iteratively, that is, given a routing template and a capacity allocation, Scutellà [42] allows some commodities to use a second routing template in order to reduce the overall capacity allocation. The procedure is proved to be $\mathcal{NP}$-hard in [43]. Mattia [34] provides a branch-and-cut algorithm for robust network design with dynamic routing together with a computational comparison to the static version.

As an alternative to these $\mathcal{NP}$-hard approaches, Ouorou and Vial [40] and Babonneau et al. [6] only recently apply directly the affine recourse from [12] to network design problems using particular uncertainty sets. The resulting restrictive routing scheme is refereed to as *affine routing* in the following. We will introduce affine routing as a generalization of static routing. In this context affine routing provides an alternative in between static and dynamic routing yielding tractable robust counterparts in contrast to the schemes used in [7, 43].

The contributions of this paper consists of a theoretical and empirical study of network design under the affine routing principle for general polyhedral demand uncertainty sets $\mathcal{D}$. We embed affine routing into the context of two-stage network design with recourse and compare it to its natural counterparts, static and dynamic routings. Section 2 introduces the mathematical models and defines formally static, affine and dynamic routings. In Section 3 we show that, when $\mathcal{D}$ is full-dimensional, affine routings decompose into a combination of cycles and paths, that is, a routing template for every commodity can be affinely adjusted using a set of cycles, whenever a different commodity is perturbed within the feasible demand region. We describe then necessary and sufficient conditions on $\mathcal{D}$ under which affine routing is equivalent to static or dynamic routing. As a bi-product, we obtain that static and dynamic routings are equivalent under certain assumptions on $\mathcal{D}$. We show that dominated demand vectors are not automatically supported by affine solutions in contrast to static and dynamic solutions. In particular very small demand scenarios have to be included in the uncertainty set. Finally, Section 4 presents numerical comparisons of dynamic, affine and static routings carried out on instances from SNDlib, see [39]. It turns out that for these instances the affine routing principle is numerically very close to the dynamic second stage recourse rule. In fact, it provides enough flexibility to approximate the cost for optimal two-stage solutions with full flexibility.

## 2 Robust network design with recourse

We are given a directed graph $G = (V, A)$ and a set of commodities $K$. A commodity $k \in K$ has source $s(k) \in V$, destination $t(k) \in V$, and demand value $d^k \geq 0$. A *flow* for $k$ is a vector $f^k \in \mathbb{R}^A_+$ satisfying:

$$\sum_{a\in\delta^+(v)} f^k_a - \sum_{a\in\delta^-(v)} f^k_a = d^k \psi_{vk} \quad \text{for all } v \in V, \tag{1}$$

where $\delta^+(v)$ and $\delta^-(v)$ denote the set of outgoing arcs and incoming arcs at node $v$, respectively. For node $v \in V$ and commodity $k \in K$ we set $\psi_{vk} := 1$ if $v = s(k)$, $\psi_{vk} := -1$ if $v = t(k)$, and $\psi_{vk} := 0$ else. Flows are non-negative. A *multi-commodity flow* is a collection of flows, one for each commodity in $K$. A *circulation* (or cycle-flow) is a vector $g \in \mathbb{R}^A$ satisfying

$$\sum_{a\in\delta^+(v)} g_a - \sum_{a\in\delta^-(v)} g_a = 0 \quad \text{for all } v \in V. \tag{2}$$

A circulation is not necessarily non-negative. A value $g_a < 0$ can be seen as a flow from the head of arc $a$ to its tail. We call a circulation $g$ *non-negative* if $g \geq 0$ and *positive* if additionally $g \neq 0$.

Notice that any two flows $\hat{f}^k, f^k$ for $k$ only differ by a circulation, that is, there always exists a circulation $g$ such that $\hat{f}^k = f^k + g$.

In many practical situations, the demand vector $d \in \mathbb{R}_+^K$ is uncertain. In the sequel we assume that $d \in \mathcal{D} \subset \mathbb{R}^K$ where $\mathcal{D}$ is a bounded convex set. We often call $\mathcal{D}$ the *uncertainty set*. Any $d \in \mathcal{D}$ is said to be a realization of the demand. A *routing* is a function $f : \mathcal{D} \to \mathbb{R}_+^{A \times K}$ that assigns a multi-commodity flow to every realization of the demand. We say that $f$ *serves* $\mathcal{D}$ and call $f$ a *dynamic routing* if there is no further restriction on the routing. A capacity allocation $x \in \mathbb{R}_+^A$ is said to *support* the set $\mathcal{D}$ if there exists a routing $f$ serving $\mathcal{D}$ such that for every $d \in \mathcal{D}$ the corresponding multi-commodity flow $f(d)$ is not exceeding the arc-capacities given by $x$. Robust network design now aims at providing the cost minimal capacity allocation supporting $\mathcal{D}$. In this respect, robust network design is a two-stage robust program with recourse, following the more general framework described by [12]. The capacity design has to be fixed in the first stage, and, observing a demand realization $d \in \mathcal{D}$, we are allowed to adjust the routing $f(d)$ in the second stage. The problem can be written as the following (infinite) linear program denoted by $(RND)$ in the following:

$$
(RND) \qquad \begin{aligned}
\min \quad & \sum_{a \in A} \kappa_a x_a && && (3) \\
\text{s.t.} \quad & \sum_{a \in \delta^+(v)} f_a^k(d) - \sum_{a \in \delta^-(v)} f_a^k(d) &= d^k \psi_{vk}, & \quad v \in V, k \in K, d \in \mathcal{D} && (4) \\
& \sum_{k \in K} f_a^k(d) &\le x_a, & \quad a \in A, d \in \mathcal{D} && (5) \\
& f_a^k(d) &\ge 0, & \quad a \in A, k \in K, d \in \mathcal{D} && (6) \\
& x_a &\ge 0, & \quad a \in A,
\end{aligned}
$$

where $\kappa_a \in \mathbb{R}_+$ is the cost for installing one unit of capacity on arc $a \in A$. Ben-Tal et al. [12] point out that allowing for arbitrary recourse very often makes robust optimization problems intractable. In fact, this is true for two-stage robust network design with free recourse. It is known that already deciding whether a given capacity vector $x$ supports $\mathcal{D}$ is $\mathcal{NP}$-complete for general polytopes $\mathcal{D}$, see [20, 28]. It follows from this $\mathcal{NP}$-completeness result that it is impossible (unless $\mathcal{P} = \mathcal{NP}$) to derive a compact formulation for $(RND)$ given a general uncertainty polytope $\mathcal{D}$ if there is no restriction on the second stage routing decision, contrasting with the reformulation discussed in Section 3.1. Using a branch-and-cut approach to solve $(RND)$ based on Benders decomposition ([14]), Mattia [34] shows how to solve the $\mathcal{NP}$-hard separation problem for robust metric inequalities ([30, 36]) using bilevel and mixed integer programs.

Most authors ([2, 4, 9, 32, 35, 37], among others) use a simpler version of $(RND)$ introducing a restriction on the second stage recourse known as *static routing* (also called oblivious routing). Each component $f^k : \mathcal{D} \to \mathbb{R}_+^A$ is forced to be a linear function of $d^k$:

$$f_a^k(d) := y_a^k d^k \qquad a \in A, k \in K, d \in \mathcal{D}. \tag{7}$$

Notice that by (7) the flow for $k$ is not changing if we perturb the demand for $h \neq k$. By combining (4) and (7) it follows that the multipliers $y \in \mathbb{R}_+^{A \times K}$ define a multi-commodity (percentage) flow. For every $k \in K$, the vector $y^k \in \mathbb{R}_+^A$ satisfies (1) setting $d^k = 1$. The flow $y$ is called a *routing template* since it decides, for every commodity, which paths are used to route the demand and what is the percental splitting among these paths. The routing template has to be used by all demand scenarios $d \in \mathcal{D}$ under the static routing scheme.

Ben-Tal et al. [12] introduce Affine Adjustable Robust Counterparts restricting the recourse to be an affine function of the uncertainties. Applying this framework to $(RND)$ means restricting $f^k$ to be an affine function of *all* components of $d$ giving

$$f_a^k(d) := f_a^{0k} + \sum_{h \in K} y_a^{kh} d^h \ge 0, \qquad a \in A, k \in K, d \in \mathcal{D}, \tag{8}$$