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Exact solutions of nonlinear partial differential equations by the method of group foliation reduction

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EXACT SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS BY THE METHOD OF GROUP FOLIATION REDUCTION

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Abstract. A novel symmetry method for finding exact solutions to nonlinear PDEs is illustrated by applying it to a semilinear reaction-diffusion equation in multi-dimensions. The method is based on group foliation reduction and employs a separation ansatz to solve an equivalent first-order group foliation system whose independent and dependent variables respectively consist of the invariants and differential invariants of a given one-dimensional group of point symmetries for the reaction-diffusion equation. With this method, solutions of the reaction-diffusion equation are obtained in an explicit form, including group-invariant similarity solutions and travelling-wave solutions, as well as dynamically interesting solutions that are not invariant under any of the point symmetries admitted by this equation.

1. Introduction

The construction of group foliations using admitted point symmetry groups for nonlinear partial differential equations (PDEs) is originally due to Lie and Vessiot and was revived in its modern form by Ovsiannikov [1]. In general a group foliation converts a given nonlinear PDE into an equivalent first-order PDE system, called the group-resolving equations, whose independent and dependent variables respectively consist of the invariants and differential invariants of a given one-dimensional group of point symmetry transformations. Each solution of the group-resolving equations geometrically corresponds to an explicit one-parameter family of exact solutions of the original nonlinear PDE, such that the family is closed under the given one-dimensional symmetry group acting in the solution space of the PDE.

Because a group foliation contains all solutions of the given nonlinear PDE, ansatzes or differential-algebraic constraints must be used to reduce the group-resolving equations into an overdetermined system for the purpose of obtaining explicit solutions. Compared with classical symmetry reduction [2, 3], a main difficulty to-date has been how to find effective, systematic ansatzes that lead to useful reductions.

An important step toward overcoming this difficulty has been taken in recent work [4, 5] on finding exact solutions to semilinear wave equations and heat equations with power nonlinearities. Specifically, this work demonstrates that the group-resolving equations for such nonlinear PDEs have solutions arising from a simple separation ansatz in terms of the group-invariant variables. Through this ansatz, many explicit solutions to the nonlinear PDE are easily found, whose form would not be readily obvious just by trying simple direct ansatzes using the original independent and dependent variables in the nonlinear PDE, or by simply writing down the form for classical group-invariant solutions. In particular, some of these solutions are not invariant under any of the point symmetries of the nonlinear PDE and thus fall completely outside of classical symmetry reduction

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(while others coincide with explicit group-invariant solutions). Most importantly for applications, many of the solutions also have interesting analytical properties related to critical dynamics, blow-up behaviour, asymptotic behaviour and attractors.

We will illustrate this group-foliation reduction method by applying it to obtain explicit exact solutions for the semilinear radial reaction-diffusion equation

$$u_t = u_{rr} + (n-1)r^{-1}u_r + (p - ku^q)u, \quad k = \pm 1, \quad p = \text{const.} \tag{1}$$

for $u(t,r)$, with a nonlinearity power $q \neq 0, -1$, where r denotes the radial coordinate in $n > 1$ dimensions or the half-line coordinate in $n = 1$ dimension. The symmetry structure of this reaction-diffusion equation is given by [6]

$$\text{time translation} \quad \mathbf{X}_1 = \partial/\partial t \quad \text{for all } n, q, p, \tag{2}$$

$$\text{scaling} \quad \mathbf{X}_2 = 2t\partial/\partial t + r\partial/\partial r - (2/q)u\partial/\partial u \quad \text{only for } p = 0, \tag{3}$$

$$\text{space translation} \quad \mathbf{X}_3 = \partial/\partial r \quad \text{only for } n = 1, \tag{4}$$

where $\mathbf{X}$ is the infinitesimal generator of a one-parameter group of point transformations acting on (t, r, u). For constructing a group foliation, it is natural to use the time translation generator (2), since this is the only point symmetry admitted for all cases of the parameters $n \geq 1$, $q \neq 0$, p.

In Sec. 2, we first set up the system of group-resolving equations given by the time-translation symmetry (2) for the reaction-diffusion equation (1), which uses the invariants and differential invariants of the symmetry generator $\mathbf{X}_1$ as the independent and dependent variables in the system. We next state the form required for solutions of the group-resolving system to correspond to group-invariant solutions of the reaction-diffusion equation (1) with respect to the point symmetries generated by $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$.

In Sec. 3, we explain the separation ansatz for directly reducing the system of group-resolving equations. This reduction yields an overdetermined system of differential-algebraic equations which can be readily solved by computer algebra. We present the explicit solutions of these equations and then we derive the resulting exact solutions of the reaction-diffusion equation. These solutions include explicit similarity solutions in the case $p = 0$, and explicit travelling wave solutions in addition to an explicit non-invariant solution in the case $n = 1$.

In Sec. 4, we show how the success of the reduction ansatz can be understood equivalently as constructing partially-invariant subspaces for a nonlinear operator that arises in a natural way from the structure of the group-resolving equations. This important observation puts our method on a wider mathematical foundation within the general theory of invariant subspaces developed by Galaktionov [7].

Finally, we make some general concluding remarks in Sec. 5.

2. Group-resolving equations and symmetries

To proceed with setting up the time-translation group foliation for the reaction-diffusion equation (1), we first write down the invariants (in terms of t, r, u)

$$x = r, \quad v = u, \tag{5}$$

satisfying $\mathbf{X}_1 x = \mathbf{X}_1 v = 0$, and the differential invariants (in terms of u_t, u_r)

$$G = u_t, \quad H = u_r, \tag{6}$$

satisfying $\mathbf{X}_1^{(1)} G = \mathbf{X}_1^{(1)} H = 0$ where $\mathbf{X}_1^{(1)}$ is the first-order prolongation of the generator (2). Here x and v are mutually independent, while G and H are related by equality of mixed r, t derivatives on u_t and u_r, which gives

$$D_r G = D_t H \tag{7}$$

where D_r, D_t denote total derivatives with respect to r, t. In addition, v, G, H are related through the reaction-diffusion equation (1) by

$$G - r^{1-n} D_r(r^{n-1} H) = (p - kv^q)v. \tag{8}$$

Now we put $G = G(x, v)$, $H = H(x, v)$ into equations (7) and (8) and use equation (5) combined with the chain rule to arrive at a first-order PDE system

$$G_x + HG_v - GH_v = 0, \tag{9}$$

$$G - (n-1)H/x - H_x - HH_v = (p - kv^q)v, \tag{10}$$

with independent variables x, v, and dependent variables G, H. These PDEs are called the *time-translation-group resolving system* for the reaction-diffusion equation (1).

The respective solution spaces of equation (1) and system (9)–(10) are related by a group-invariant mapping that is defined through the invariants (5) and differential invariants (6).

Lemma 1. *Solutions $(G(x, v), H(x, v))$ of the time-translation-group resolving system (9)–(10) are in one-to-one correspondence with one-parameter families of solutions $u(t, r, c)$ of the reaction-diffusion equation (1) satisfying the translation-invariance property*

$$u(t + \epsilon, r, c) = u(t, r, \tilde{c}(\epsilon, c)) \tag{11}$$

where $\tilde{c}(0, c) = c$ in terms of an arbitrary constant c and parameter ϵ, such that

$$u_t = G(r, u), \quad u_r = H(r, u) \tag{12}$$

constitutes a consistent pair of parametric first-order ODEs whose integration constant is c.

We now examine the relationship between the symmetry structure of the reaction-diffusion equation (1) and the symmetry structure inherited by the time-translation-group resolving system (9)–(10).

Firstly, through the identifications defined by the variables (5)–(6), the prolongation of any point symmetry generator $\mathbf{X} = a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + a_3\mathbf{X}_3$ of equation (1) has a natural projection to a point symmetry generator $\mathbf{Y} = a_2\mathbf{X}_2^{(1)} + a_3\mathbf{X}_3^{(1)}$ modulo $\mathbf{X}_1^{(1)}$ of system (9)–(10). The time-translation $\mathbf{X}_1$ thus gets annihilated by this projection, i.e. $\mathbf{Y}_1 = 0$, while the scaling $\mathbf{X}_2$ and the space-translation $\mathbf{X}_3$ respectively project to

$$\mathbf{Y}_2 = x\partial/\partial x - (2/q)v\partial/\partial v - 2(1 + 1/q)G\partial/\partial G - (1 + 2/q)H\partial/\partial H \quad \text{when } p = 0, \tag{13}$$

and

$$\mathbf{Y}_3 = \partial/\partial x \quad \text{when } n = 1. \tag{14}$$

Secondly, with respect to these inherited symmetries (13) and (14), the system (9)–(10) has a reduction to ODEs yielding solutions where (G, H) is invariant respectively under scalings

$$x \to \lambda x, \quad v \to \lambda^{-2/q} v \quad \text{when } p = 0, \tag{15}$$

and under translations

$$x \to x + \epsilon \quad \text{when } n = 1. \tag{16}$$

Thus, translation-invariant solutions have the form

$$(G, H) = (g(v), h(v)) \tag{17}$$

satisfying the ODE system

$$(h/g)' = 0, \quad g - hh' = (p - kv^q)v. \tag{18}$$

Scaling-invariant solutions have the form

$$(G, H) = (x^{-2-2/q}g(V), x^{-1-2/q}h(V)), \quad V = vx^{2/q} \tag{19}$$

satisfying the ODE system

$$((h + 2V/q)/g)' = -2/g, \quad g + (2 - n + 2/q)h - (h + 2V/q)h' = -kV^{q+1}. \tag{20}$$

Integration of the parametric ODEs (12) for such solutions (17)–(18) and (19)–(20) leads to the following two correspondence results.

Lemma 2. *In the case $n = 1$, there is a one-to-one correspondence between solutions of the translation-group resolving system (9)–(10) with the invariant form (17) and one-parameter families of travelling-wave solutions of the reaction-diffusion equation (1) given by the group-invariant form $u = f(\xi)$ where, modulo time-translations $t \to t + c$, the variable $\xi = r - t/a$ is an invariant of the translation symmetry $\mathbf{X} = a\partial/\partial t + \partial/\partial r = a\mathbf{X}_1 + \mathbf{X}_3$ in terms of some constant a (determined by ODE system (18)).*

Lemma 3. *In the case $p = 0$, there is a one-to-one correspondence between solutions of the translation-group resolving system (9)–(10) with the invariant form (19) and one-parameter families of similarity solutions of the reaction-diffusion equation (1) given by the group-invariant form $u = r^{-2/q}f(\xi)$ where, modulo time-translations $t \to t + c$, the variable $\xi = t/r^2$ is an invariant of the scaling symmetry $\mathbf{X} = 2t\partial/\partial t + r\partial/\partial r - (2/q)u\partial/\partial u = \mathbf{X}_2$.*

Furthermore, in all cases, static solutions $u(r)$ of the reaction-diffusion equation correspond to solutions of the translation-group resolving system with $G = 0$. Hereafter we will be interested only in solutions with $G \neq 0$, corresponding to dynamical solutions of the reaction-diffusion equation.

3. Main results

To find explicit solutions of the group foliation system (9)–(10) for $(G(x, v), H(x, v))$, we will make use of the same general homogeneity features utilized in Refs. [4, 5]. First, the non-derivative terms $(p - ku^q)u$ in the reaction-diffusion equation (1) appear only as an inhomogeneous term in equation (10). Second, in both equations (9) and (10) the linear terms involve no derivatives with respect to v. Third, the nonlinear terms in the homogeneous equation (9) have the skew-symmetric form $HG_v - GH_v$, while HH_v is the only nonlinear term appearing in the non-homogeneous equation (10). Based on these features, this system can be expected to have solutions given by the separable power form

$$G = g_1(x)v + g_2(x)v^a, \quad H = h_1(x)v + h_2(x)v^a, \quad a \neq 1. \tag{21}$$

For such a separation ansatz (21), the linear terms G_x, G, H/x, H_x in equations (9) and (10) will contain the same powers v, v^a that appear in both G and H, and moreover the nonlinear term $HG_v - GH_v$ in the homogeneous equation (9) will produce only the power v^a due to the identities $v^a(v)_v - v(v^a)_v = (a - 1)v^a$ and $v(v)_v - v(v)_v = v^a(v^a)_v - v^a(v^a)_v = 0$. Thus, equation (9) can be satisfied by having the coefficients of v and v^a separately vanish. Similarly the nonlinear term HH_v in the non-homogeneous equation (10) will only yield the powers v, v^a, v^{2a-1}. Since we have $a \neq 1$ and $q \neq 0$, equation (10) can be satisfied by again having the coefficients of v and v^a separately vanish and by also having the term containing v^{2a-1} balance the inhomogeneous term kv^{q+1}. In this fashion we find that equations (9) and (10) reduce to an overdetermined