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Symmetry analysis and exact solutions of semilinear heat flow in multi-dimensions

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SYMMETRY ANALYSIS AND EXACT SOLUTIONS OF SEMILINEAR HEAT FLOW IN MULTI-DIMENSIONS

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ABSTRACT. A symmetry group method is used to obtain exact solutions for a semilinear radial heat equation in $n > 1$ dimensions with a general power nonlinearity. The method involves an ansatz technique to solve an equivalent first-order PDE system of similarity variables given by group foliations of this heat equation, using its admitted group of scaling symmetries. This technique yields explicit similarity solutions as well as other explicit solutions of a more general (non-similarity) form having interesting analytical behavior connected with blow up and dispersion. In contrast, standard similarity reduction of this heat equation gives a semilinear ODE that cannot be explicitly solved by familiar integration techniques such as point symmetry reduction or integrating factors.

1. Introduction

In the study of nonlinear partial differential equations (PDEs), similarity solutions are important for the understanding of asymptotic behaviour and attractors, critical dynamics, and blow-up behaviour. Such solutions are characterized by a scaling homogeneous form arising from invariance of a PDE under a point symmetry group of scaling transformations that act on the independent and dependent variables in the PDE [1, 2].

For scaling invariant PDEs that have only two independent variables, similarity solutions satisfy an ordinary differential equation (ODE) formulated in terms of the invariants of the scaling transformations. However, this ODE can often be very difficult to solve explicitly, and as a consequence, special ansatzes or ad hoc techniques may be necessary in order to obtain any solutions in an explicit form. The same difficulties occur more generally in trying to find explicit group-invariant solutions to nonlinear PDEs with other types of point symmetry groups.

An interesting example is the semilinear radial heat equation

$$u_t = u_{rr} + (n-1)r^{-1}u_r + k|u|^q u, \quad k = \pm 1 \tag{1}$$

for $u(t, r)$, which has the scaling symmetry group

$$t \to \lambda^2 t, \quad r \to \lambda r, \quad u \to \lambda^{-2/q} u, \tag{2}$$

where r denotes the radial coordinate in $n > 1$ dimensions. This equation describes radial heat flow with a nonlinear heat source/sink term depending on a power $q \neq 0$. The coefficient k of this term determines the stability of solutions to the initial-value problem. In particular, for $k = -1$ all smooth solutions $u(t, r)$ asymptotically approach a similarity form $u = t^{-1/q}U(r/\sqrt{t})$ exhibiting global dispersive behaviour $u \to 0$ as $t \to \infty$ for all $r \geq 0$, while for $k = 1$ some solutions $u(t, r)$ exhibit a blow-up behaviour $u \to \infty$ given by a similarity form $u = (T-t)^{-1/q}U(r/\sqrt{T-t})$ as $t \to T < \infty$ [3, 4]. In both cases, U satisfies a nonlinear ODE

$$U'' + ((n-1)\xi^{-1} - \tfrac{1}{2}k\xi)U' - \tfrac{1}{2}qkU + kU|U|^q = 0 \tag{3}$$

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with

$$\xi = \begin{cases} r/\sqrt{t}, & k=-1 \\ r/\sqrt{T-t}, & k=1 \end{cases} \tag{4}$$

which arises from the scaling invariance (2). This ODE cannot be explicitly solved by standard integration techniques [2] such as symmetry reduction or integrating factors when $n \neq 1$ and $q \neq 0$. (More specifically, ODE (3) has no point symmetries $X = \eta(\xi,U)\partial/\partial U + \zeta(\xi,U)\partial/\partial\xi$ and no quadratic first integrals $\Psi = A(\xi,U) + B(\xi,U)U' + C(\xi,U)U'^2$, as established by solving the standard determining equations [1, 2] for X and Ψ.) As a result, few exact solutions $U(\xi)$ other than the explicit constant solution $U = (q/2)^{1/q}$ are apparently known to-date.

In this paper we will obtain explicit exact solutions for the heat equation (1) by applying an alternative similarity method developed in previous work [5] on finding exact solutions to a semilinear radial wave equation with a power nonlinearity. The method uses the group foliation equations [6] associated with a given point symmetry of a nonlinear PDE. These equations consist of an equivalent first-order PDE system whose independent and dependent variables are respectively given by the invariants and differential invariants of the point symmetry transformation. In the case of a PDE with power nonlinearities, the form of the resulting group-foliation system allows explicit solutions to be found by a systematic separation technique in terms of the group-invariant variables. Each solution of the system geometrically corresponds to an explicit one-parameter family of exact solutions of the original nonlinear PDE, such that the family is closed under the given symmetry group acting in the solution space of the PDE.

In Sec. 2, we set up the group foliation system given by the scaling symmetry (2) for the heat equation (1) and explain the separation technique that we use to find explicit solutions of this system. The resulting exact solutions of the heat equation are summarized in Sec. 3. These solutions include explicit similarity solutions as well as other solutions whose form is not scaling homogeneous, and we discuss their analytical features of interest pertaining to blow-up and dispersion. Finally, we make some concluding remarks in Sec. 4.

Related work using a similar method applied to nonlinear diffusion equations appears in Ref. [7]. Group foliation equations were first used successfully in Refs. [8, 9, 10] for obtaining exact solutions to nonlinear PDEs by a different method that is applicable when the group of point symmetries of a given PDE is infinite-dimensional, compared to the example of a finite-dimensional symmetry group considered both in Ref. [5] and in the present work.

2. Symmetries and group foliation

For the purpose of symmetry analysis and finding exact solutions, it is easier to work with a slightly modified form of the heat equation (1):

$$u_t = u_{rr} + (n-1)r^{-1}u_r + ku^{q+1}, \quad k = \text{const.}, \quad q \neq 0. \tag{5}$$

In $n > 1$ dimensions, this heat equation (5) admits only two point symmetries:

$$\text{time translation} \quad \mathbf{X}_{\text{trans.}} = \partial/\partial t \quad \text{for all } q, \tag{6}$$

$$\text{scaling} \quad \mathbf{X}_{\text{scal.}} = 2t\partial/\partial t + r\partial/\partial r - (2/q)u\partial/\partial u \quad \text{for all } q \neq 0, \tag{7}$$

where $\mathbf{X}_{.}$ is the infinitesimal generator of a one-parameter group of point transformations acting on (t, r, u). There are no special powers or dimensions for which any extra point symmetries exist for equation (5), as found by a direct analysis of the symmetry determining equations.

To proceed with setting up the group foliation equations using the scaling point symmetry, we first write down the invariants and differential invariants determined by the generator (7). The simplest invariants in terms of t, r, u are given by

$$x = t/r^2, \quad v = u/r^p, \tag{8}$$

satisfying $\mathbf{X}_{\text{scal.}}x = \mathbf{X}_{\text{scal.}}v = 0$ with

$$p = -2/q. \tag{9}$$

A convenient choice of differential invariants satisfying $\mathbf{X}^{(1)}_{\text{scal.}}G = \mathbf{X}^{(1)}_{\text{scal.}}H = 0$ for $G(t, r, u_t)$ and $H(t, r, u_r)$ consists of

$$G = r^{2-p}u_t, \quad H = r^{1-p}u_r, \tag{10}$$

where $\mathbf{X}^{(1)}_{\text{scal.}} = \mathbf{X}_{\text{scal.}} - (2+2/q)u_t\partial/\partial u_t - (1+2/q)u_r\partial/\partial u_r$ is the first-order prolongation of the generator (7). Here x and v are mutually independent, while G and H are related by equality of mixed r, t derivatives on u_t and u_r, which gives

$$D_r(r^{p-2}G) = D_t(r^{p-1}H) \tag{11}$$

where D_r, D_t denote total derivatives with respect to r, t. Furthermore, v, G, H are related through the heat equation (5) by

$$r^{p-2}G - D_r(r^{p-1}H) = r^{p-2}((n-1)H + kv^{q+1}). \tag{12}$$

Now we put $G = G(x, v)$, $H = H(x, v)$ into equations (11) and (12) and use equation (8) combined with the chain rule to arrive at a first-order PDE system

$$(p-2)G - pvG_v - 2xG_x - H_x + HG_v - GH_v = 0, \tag{13}$$

$$G - (p+n-2)H + pvH_v + 2xH_x - HH_v = kv^{q+1}, \tag{14}$$

with independent variables x, v, and dependent variables G, H. These PDEs are called the *scaling-group resolving system* for the heat equation (5).

The respective solution spaces of equation (5) and system (13)–(14) are related by a group-invariant mapping that is defined through the invariants (8) and differential invariants (10). In particular, the map $(G, H) \to u$ is given by integration of a consistent pair of parametric first-order ODEs

$$u_t = r^{p-2}G(t/r^2, u/r^p), \quad u_r = r^{p-1}H(t/r^2, u/r^p) \tag{15}$$

whose general solution will involve a single arbitrary constant. The inverse map $u \to (G, H)$ can be derived in the same way as shown in Ref. [5] for the wave equation, which gives the following correspondence result.

Lemma 1. *Solutions $(G(x, v), H(x, v))$ of the scaling-group resolving system (13)–(14) are in one-to-one correspondence with one-parameter families of solutions $u(t, r, c)$ of the heat equation (5) satisfying the scaling-invariance property*

$$\lambda^{-p}u(\lambda^2 t, \lambda r, c) = u(t, r, \tilde{c}(\lambda, c)) \tag{16}$$

where $\tilde{c}(1, c) = c$ in terms of an arbitrary constant c.

This correspondence leads to an explicit characterization of similarity solutions of the heat equation (5) in terms of a condition on solutions of the scaling-group resolving system (13)–(14). Consider any one-parameter family of solutions

$$u(t, r) = r^p v, \quad v = V(x, c), \tag{17}$$

having a scaling-homogeneous form, where

$$4x^2V'' - (1 + (2p + n - 4)2x)V' + p(p + n - 2)V + kV^{q+1} = 0 \tag{18}$$

is the ODE given by reduction of PDE (5). ¿From relation (10) we have

$$G(x, V(x, c)) = V'(x, c), \quad H(x, V(x, c)) = pV(x, c) - 2xV'(x, c). \tag{19}$$

Next we eliminate c in terms of x and v by using the implicit function theorem on $V(x, c) - v = 0$ to express $c = C(x, v)$. Substitution of this expression into equation (19) yields

$$H + 2xG = pv \tag{20}$$

where $G = V'(x, C(x, v))$, $H = pv - 2xV'(x, C(x, v))$ are some functions of x, v. The relation (20) is easily verified to satisfy PDE (13). In addition, PDE (14) simplifies to

$$-4x^2(G_x + GG_v) + (1 + (2p + n - 4)2x)G = p(p + n - 2)v + kv^{q+1}. \tag{21}$$

We then see that the characteristic ODEs for solving this first-order PDE are precisely

$$dv/dx = G, \quad -4x^2dG/dx + (1 + (2p + n - 4)2x)G = p(p + n - 2)v + kv^{q+1}, \tag{22}$$

which are satisfied due to equations (18) and (19). Hence, we have established the following result.

Lemma 2. *There is a one-to-one correspondence between one-parameter families of similarity solutions (17) of heat equation (5) and solutions of the scaling-group resolving system (13)–(14) that satisfy the similarity relation (20).*

We now note that, under the mapping (15), static solutions $u(r)$ of the heat equation correspond to solutions of the scaling-group resolving system with $G = 0$. Consequently, hereafter we will be interested only in solutions such that $G \neq 0$, corresponding to dynamical solutions of the heat equation.

To find explicit solutions of the PDE system (13)–(14) for $G(x, v), H(x, v)$, we will exploit its following general features. First, the power nonlinearity ku^{q+1} in the heat equation appears only as an inhomogeneous term kv^{q+1} in the PDE (14). Second, in both PDEs (13) and (14) the linear terms that involve v derivatives have the scaling homogeneous form vG_v and vH_v with respect to v. Third, the nonlinear terms in the homogeneous PDE (13) have the skew-symmetric form $HG_v - GH_v$, while HH_v is the only nonlinear term appearing in the non-homogeneous PDE (14). These features suggest that this PDE system can be expected to have solutions given by the separable power form

$$G = g_1(x)v^a + g_2(x)v, \quad H = h_1(x)v^a + h_2(x)v, \quad a \neq 1. \tag{23}$$

For such an ansatz, we readily see that the linear derivative terms G_x, H_x, vG_v, vH_v in each PDE (13) and (14) will contain the same powers v, v^a that appear in both G and H, and moreover the nonlinear term $HG_v - GH_v$ in the homogeneous PDE (13) will produce only the power v^a due to the identities $v^a(v)_v - v(v^a)_v = (a - 1)v^a$ and $v(v)_v - v(v)_v = v^a(v^a)_v - v^a(v^a)_v = 0$. Similarly we see that the nonlinear term HH_v in the non-homogeneous PDE (14) will only yield the powers v, v^a, v^{2a-1}. Since we have $a \neq 1$ and $q \neq 0$, the inhomogeneous term kv^{q+1} must therefore balance one of the powers v^{2a-1} or v^a.

In the case when we balance $q + 1 = a$, the terms containing $v^a = v^{q+1}$ and v^{2a-1} in PDE (14) immediately yield

$$h_1 = 0, \quad g_1 = k. \tag{24}$$