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On the Length of the Primal-Dual Path in Moreau-Yosida-based Path-following for State Constrained Optimal Control: Analysis and Numerics †

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Abstract

We derive a-priori estimates on the length of the primal-dual path that results from a Moreau-Yosida approximation of the feasible set for state constrained optimal control problems. These bounds depend on the regularity of the state and the dimension of the problem. Comparison with numerical results indicates that these bounds are sharp and are attained for the case of a single active point.
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1 Introduction

In recent years, path-following methods based on the Moreau-Yosida (or quadratic penalty) regularization of state constrained problems have received considerable attention. While general results on the convergence of this method can be derived under very mild assumptions, deriving estimates on the length of corresponding homotopy path, the "primal-dual path", and its asymptotic behavior is more delicate. In particular, numerical experience shows that this asymptotic behavior varies from problem to problem.

The purpose of this note is twofold. First, we present a-priori error estimates on the order of convergence of the primal-dual path that depend on the dimensionality of the problem and the smoothness of the solution. In comparison to the estimates that were derived in [9] we obtain an improvement in the rate, compared to [3] our results are based on a considerably weaker set of assumptions.

Second, we try to develop an understanding on the principles that govern the rate of convergence of the primal dual path. This will be accomplished by comparison of numerical and theoretical results. It will turn out that the topology of the active set plays a decisive role for the rate of convergence. For the "worst case",

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namely the case that the active set is a single touch point, our theoretical estimates coincide with the numerical observations.

To render the discussion concrete, we consider the primal-dual path-following method for a state constrained model problem in optimal control. The techniques presented here are, however, applicable in a much broader context. The main idea is to replace the problem:

$$\begin{aligned} \min_{y\in H^2(\Omega),u\in L^2(\Omega)} \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} & \\ \text{subject to} \quad -\Delta y - u = 0 \text{ in } \Omega & \\ y = 0 \text{ in } \partial\Omega & \\ y \le \psi \text{ in } \Omega & \end{aligned} \tag{1}$$

(here Ω is a smoothly bounded domain in $\mathbb{R}^d$ for $d = 1, 2, 3$, $y_d \in L_2(\Omega)$, and ψ is a smooth, strictly positive function on $\overline{\Omega}$), by a family of problems

$$\begin{aligned} \min_{y\in H^2(\Omega),u\in L^2(\Omega)} \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|\max(y-\psi,0)\|^2_{L^2(\Omega)} & \\ \text{subject to} \quad -\Delta y - u = 0 \text{ in } \Omega & \\ y = 0 \text{ in } \partial\Omega & \end{aligned} \tag{2}$$

and consider a sequence of solutions $x_\gamma := (y_\gamma, u_\gamma)$ of (2). It has been shown in [4] that this sequence converges to the original solution $x_* := (y_*, u_*)$ of (1) as γ tends to infinity.

Practical algorithms use a semi-smooth Newton method to solve discretizations of the subproblems (2) approximately or exactly. For this purpose the first order necessary conditions are derived for (2) which assert existence of an adjoint state $p_\gamma \in H^2(\Omega)$ such that

$$\begin{aligned} y_\gamma - y_d + \gamma\max(y_\gamma - \psi, 0) - \Delta p_\gamma &= 0 \text{ in } \Omega \\ p_\gamma &= 0 \text{ in } \partial\Omega \end{aligned} \tag{3}$$

$$\alpha u_\gamma - p_\gamma = 0 \text{ in } \Omega \tag{4}$$

$$\begin{aligned} -\Delta y_\gamma - u_\gamma &= 0 \text{ in } \Omega \\ y_\gamma &= 0 \text{ in } \partial\Omega. \end{aligned} \tag{5}$$

This can be compared with the first order necessary conditions for the original problem, which state existence of a measure valued Lagrangian multiplier $m \in \mathcal{M}(\overline{\Omega})$ and an adjoint state $p_* \in W^{1,q'}(\Omega)$ ($q' < d/(d-1)$), such that

$$\begin{aligned} y_* - y_d + m - \Delta p_* &= 0 \text{ in } \Omega \\ p_* &= 0 \text{ in } \partial\Omega \\ -\Delta y_* - u_* &= 0 \text{ in } \Omega \\ y_* &= 0 \text{ in } \partial\Omega \\ \alpha u_* - p_* &= 0 \text{ in } \Omega \\ m \ge 0, \quad y_* \le \psi, \quad \langle m, y_* - \psi\rangle_{\mathcal{M}(\overline{\Omega})\times C(\overline{\Omega})} &= 0 \text{ in } \overline{\Omega}. \end{aligned}$$

We observe that the function $\gamma \max(y_\gamma - \psi, 0)$ plays the role of m in the regularized setting.

Elimination of u_γ from (3)–(5) yields the system

$$F(x;\gamma) := \left\{ \begin{array}{rcl} y - y_d + \gamma \max(y-\psi,0) - \Delta p &=& 0 \text{ in } \Omega \\ p &=& 0 \text{ in } \partial\Omega \\ -\Delta y - \alpha^{-1} p &=& 0 \text{ in } \Omega \\ y &=& 0 \text{ in } \partial\Omega, \end{array} \right. \tag{6}$$

which can be tackled by a semi-smooth Newton method as shown in [4].

2 Analysis of the Length of the Primal-Dual Path

In constrained optimization and in particular in state constrained optimal control (c.f. e.g. [1]) the existence of a strictly feasible point (a Slater point) is a standard assumption for the existence of Lagrange multipliers. In state constrained optimal control this assumption is used to show that the corresponding Lagrange multipliers are positive measures. We will assume existence of a Slater point throughout the paper:

Assumption 2.1. Assume that there is a constant $e > 0$, such that

$$\psi - \breve{y} > e \text{ on } \Omega$$

for some pair $(\breve{y}, \breve{u})$ that satisfies the state equation.

Existence of a strictly feasible point and smoothness of the state variable y will allow us to bound the length of the primal-dual path by a power of γ^{-1}, which is clearly a stronger result that mere convergence of the primal-dual path for $\gamma \to \infty$.

2.1 A-priori bounds for the Constraint Violation in L^1

In the following, denote by y_γ^+ the function $\max(y-\psi, 0)$. Our first aim is to show that $\gamma \left\| y_\gamma^+ \right\|_{L^1}$ is bounded uniformly for $\gamma \to \infty$. The following technique is well established by now, and used in various contexts (cf. e.g. [2, 8, 5]).

Lemma 2.2. *The expression* $\gamma \left\| y_\gamma^+ \right\|_{L^1}$ *is uniformly bounded for* $\gamma \to \infty$.

Proof. Let S be the solution operator of the PDE, i.e., the control to state mapping. We test (3) and (4) with a feasible direction (Sv, v) from the optimal control problem, and add them (taking into account that $\langle -\Delta p_\gamma, Sv\rangle = \langle p_\gamma, v\rangle$) to obtain

$$\langle u_\gamma, v\rangle + \langle y_\gamma - y_d, Sv\rangle + \gamma \langle y_\gamma^+, Sv\rangle = 0 \qquad \forall v \in L^2(\Omega).$$

Inserting $v := \breve{u}, S\breve{u} = \breve{y}$ we obtain

$$\alpha\langle u_\gamma, \breve{u}\rangle + \langle y_\gamma - y_d, \breve{y}\rangle - \gamma\langle y_\gamma^+, \breve{y}\rangle = 0.$$