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Solving Muller Games via Safety Games*

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Abstract. We show how to transform a Muller game with n vertices into a safety game with $(n!)^3$ vertices whose solution allows to determine the winning regions of the Muller game and a winning strategy for one player.

1 Introduction

Infinite two-player games are a powerful tool in the automated verification and synthesis of non-terminating systems that have to interact with an antagonistic environment. There are also deep connections between infinite games and logical formalisms like fixed-point logics or automata on infinite objects. In such a game, two players move a token through a finite graph, thereby constructing a play which is an infinite path. The winner is determined by a winning condition, which partitions the set of infinite paths in a graph into those that are winning for Player 0 and those that are winning for Player 1. Typically, the winner of a play is only determined after infinitely many steps.

Nevertheless, in some cases it is possible to give a criterion to define a finite-duration variant of an infinite game. Such a criterion stops a play after a finite number of steps and then declares a winner based on the finite play constructed thus far. It is called sound if Player 0 has a winning strategy for the infinite-duration game if and only if Player 0 has one for the finite-duration game.

It is easy to see that there is a sound criterion for positionally determined games: the players move the token through the arena until a vertex is visited for the second time. An infinite play can then be obtained by assuming that the players continue to play the loop that they have constructed, and the winner of the finite play is declared to be the winner of this infinite continuation.

For parity games (say, min-parity), Bernet, Janin, and Walukiewicz [1] gave another sound criterion based on the following observation: let n_c be the number of vertices with priority c. If a play visits $n_c + 1$ vertices with odd priority c without visiting a smaller even priority in between, then the play has closed a loop which is losing for Player 0, assuming it is traversed from now on ad infinitum. However, no positional winning strategy can allow such a loop. Thus, Player 0 can prove that she has a winning strategy by allowing a play to visit an odd priority c at most n_c times without seeing a smaller even priority in between. This condition can be turned into a safety game whose solution allows to determine the winning regions of the parity game and a winning strategy for one of the players.

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In games that are not positionally determined, the situation gets more interesting since a player might have to pick different successors when a vertex is visited several times. Therefore, the players have to play longer before the play can be stopped and analyzed. Previous work considers Muller games which are of the form $(\mathcal{A}, \mathcal{F}_0, \mathcal{F}_1)$, where $\mathcal{A}$ is a finite arena and $(\mathcal{F}_0, \mathcal{F}_1)$ is a partition of the set of loops in the arena. Player i wins a play if the set of vertices visited infinitely often is in $\mathcal{F}_i$. Muller winning conditions are able to express all ω-regular winning conditions and subsume all other winning conditions that depend only on the infinity set of a play (e.g., Büchi, co-Büchi, parity, Rabin, or Streett conditions).

To give a sound criterion for Muller games, McNaughton [7] defined for every loop $F \in \mathcal{F}_0 \cup \mathcal{F}_1$ a scoring function Sc_F that keeps track of the number of times the set F was visited entirely (not necessarily in the same order) since the last visit of a vertex that is not in F. In an infinite play, the set of vertices seen infinitely often is the unique set F such that Sc_F tends to infinity after being reset to 0 only a finite number of times.

McNaughton proved the following criterion to be sound [7]: stop a play as soon as for some set F a score of $|F|! + 1$ is reached, and declare the winner to be the Player i such that $F \in \mathcal{F}_i$. However, it can take a large number of steps for a play to reach a score of $|F|! + 1$, as scores may increase slowly or be reset to 0. It can be shown that a play can be stopped by this criterion after at most $\prod_{j=1}^{|\mathcal{A}|}(j!+1)$ steps and there are examples in which it takes at least $\frac{1}{2}\prod_{j=1}^{|\mathcal{A}|}(j!+1)$ steps before the criterion declares a winner.

Also, a game reduction from Muller games to parity games provides another sound criterion. The reduction constructs a parity game of size $|\mathcal{A}| \cdot |\mathcal{A}|!$, and since parity games are positionally determined, a winner can be declared after the players have constructed a loop in the parity game. This gives a sound criterion that stops a play after at most $|\mathcal{A}| \cdot |\mathcal{A}|! + 1$ steps.

Both results were improved by showing that stopping a play after a score of 3 is reached for the first time is sound [2]. This criterion stops a play after at most $3^{|\mathcal{A}|}$ steps, and there are examples where this number of steps is necessary. The result is proven by constructing a winning strategy for Player i that bounds the opponent's scores by 2, provided the play starts in the winning region of Player i. Such a strategy ensures that Player i is the first to achieve a score of 3, as not all scores can be bounded. Thus, to determine the winner of a Muller game, it suffices to solve a finite reachability game in a tree of height $3^{|\mathcal{A}|}$.

However, this game only allows to determine the winner, but does not yield winning strategies, as each play ends after a bounded number of steps. We overcome this drawback by exploiting the existence of strategies that bound the losing player's scores. This implies that the winner of a Muller game can also be determined by solving a safety game. In this game, the scores of Player 1 are kept track of and Player 0 wins, if her opponent never reaches a score of 3. In this work, we analyze this safety game and show that one can turn the winning region of the player that has to bound the scores of her opponent into a finite-state winning strategy for her in the Muller game.

The size of the resulting safety game (and, thus, also the size of the finite-state winning strategy) is at most $(|\mathcal{A}|!)^3$. This is only polynomially larger than the parity game of size $|\mathcal{A}| \cdot |\mathcal{A}|!$ constructed in the game reduction mentioned above. Although our safety game is polynomially larger than the parity game, it is simpler and faster to solve than the latter.

The scores induce a partial order on the positions of the safety game. We also prove that it suffices to consider the maximal elements of this order to define a finite-state winning strategy for the player that tries to bound the scores of her opponent. This antichain approach is subject to further research that should estimate how much smaller this finite-state winning strategy can be.

We want to stress that our construction is not a proper game reduction, which would provide winning strategies no matter which player wins. Here, we only obtain a winning strategy for the player trying to avoid a score of 3. If the opponent is able to reach a score of 3, then the play stops immediately. Thus, not every play in the Muller game has a corresponding play in the safety game, as it is required in a game reduction. In fact, a game reduction from Muller games to safety or reachability games is impossible, as it would induce a continuous function mapping the winning plays of the Muller game to the winning plays of a safety or reachability game. Such a mapping cannot exist, since the set of winning plays of a Muller game is on a higher level of the Borel hierarchy than the set of winning plays of a safety or reachability game.

The remainder of this report is structured as follows: in Section 2 we introduce our notation, and in Section 3 we define the scoring functions for Muller games. Then, in Section 4 we show how to solve a Muller game (i.e., how to determine the winning regions and compute a winning strategy) by solving a safety game. In this context, we present an alternative way to compute a winning strategy based on antichains in Section 4.1 and discuss how to reduce the number of memory states needed to define a winning strategy in Section 4.2. Finally, Section 5 contains a brief conclusion.

2 Definitions

The power set of a set S is denoted by 2^S and $\mathbb{N}$ denotes the non-negative integers. The prefix relation on words is denoted by $\sqsubseteq$. Given a word $w = xy$, define $wy^{-1} = x$. For a non-empty word $w = w_1 \cdots w_n$, we define $\text{Last}(w) = w_n$.

An arena $\mathcal{A} = (V, V_0, V_1, E)$ consists of a finite, directed graph (V, E) without terminal vertices and a partition $\{V_0, V_1\}$ of V denoting the positions of Player 0 (drawn as circles or rectangles with rounded corners) and Player 1 (drawn as squares or rectangles). We require every vertex to have an outgoing edge to avoid the nuisance of dealing with finite plays. The size $|\mathcal{A}|$ of $\mathcal{A}$ is the cardinality of V. A loop $C \subseteq V$ in $\mathcal{A}$ is a strongly connected subset of V, i.e., for every $v, v' \in C$ there is a path from v to v' that only visits vertices in C.

A safety game $\mathcal{G} = (\mathcal{A}, F)$ consists of an arena $\mathcal{A}$ and a set $F \subseteq V$. A Muller game $\mathcal{G} = (\mathcal{A}, \mathcal{F}_0, \mathcal{F}_1)$ consists of an arena $\mathcal{A}$ and a partition $\{\mathcal{F}_0, \mathcal{F}_1\}$ of the set of loops in $\mathcal{A}$.

A play in $\mathcal{A}$ starting in $v \in V$ is an infinite sequence $\rho = \rho_0\rho_1\rho_2\ldots$ such that $\rho_0 = v$ and $(\rho_n, \rho_{n+1}) \in E$ for all $n \in \mathbb{N}$. The occurrence set $\text{Occ}(\rho)$ and infinity set $\text{Inf}(\rho)$ of ρ are given by $\text{Occ}(\rho) = \{v \in V \mid \exists n \in \mathbb{N} \text{ such that } \rho_n = v\}$ and $\text{Inf}(\rho) = \{v \in V \mid \exists^\omega n \in \mathbb{N} \text{ such that } \rho_n = v\}$. We also use the occurrence set of a finite play w, which is defined straightforwardly. The infinity set of a play is always a loop in the arena.