Hamburger Beiträge
zur Angewandten Mathematik

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Nr. 2013-08
July 2013

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Abstract. We consider balancing and model reduction by balanced truncation for infinite-dimensional linear systems. A functional analytic approach to state space transformations leading to balanced realizations is presented. These transformations can be further used to explicitly construct truncated balanced realizations. The presented approach is applicable to bounded well-posed linear systems with nuclear Hankel operator and finite-dimensional input and output space. Controllability and observability are not required.

Balanced truncation is one of the most popular methods for model reduction of asymptotically stable input-output systems of the form $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$, where A, B, C and D are matrices of suitable size. Besides preservation of asymptotical stability it also provides an a priori error bound in the $\mathcal{H}_\infty$-norm in terms of the twice the sum of the neglected Hankel singular values. The typical approach in the (numerical) determination of balanced realizations works via a state-space transformation that is constructed from the gramians P and Q of the system (see [1, Sec. 7.3] for an overview).

The probably most commonly used balancing technique in the finite-dimensional case has been introduced by Postlethwaite and Tombs in [19]: Starting from factorizations $P = RR^\top$ and $Q = SS^\top$ of the Gramians, a singular value decomposition $S^\top R = V\Sigma U^\top$ is performed. The construction $T_l = \Sigma^{-1/2}V^\top S^\top$, $T_r = RU\Sigma^{-1/2}$ and $A_b = T_l A T_r$, $B_b = T_l B$, $C_b = CT_r$, $D_b = D$ then leads to a balanced realization. If the system is minimal (i.e., it is both controllable and observable), then T_l and T_r are both square and $T_l T_r$ is the identity, whence this approach is basically a change of coordinates in the state space. In fact, this procedure can be also applied to non-minimal

This work was supported by the Klaus-Tschira-Stiftung.

systems; in this case, the multiplication with T_l and T_r eliminates the non-observable and non-controllable parts, while the input-output behavior (or, equivalently, the transfer function) is preserved. The popularity of this balancing approach is mainly based on two facts: First, there are various numerical methods for the determination of Gramians which directly provide the factors S and R instead of the Gramians themselves (see [1, Chap. 6] for an overview). Second, this balancing approach can be easily modified to directly construct the truncated balanced realization without determining the parts of A_b, B_b, C_b which are truncated anyway; this can be done by simply truncating the singular value decomposition of $S^\top R$.

The purpose of this article is to consider this balancing approach for systems with infinite-dimensional state space. Balancing and truncation has been considered in various articles [2,3,6–9,12–15]. The error bound in terms of neglected Hankel singular values has been first shown in [6] for the class of systems with square integrable impulse response, and has recently been generalized to systems with nuclear Hankel operator [8,9]. All mentioned approaches to balancing and truncation of infinite-dimensional systems have in common that they rely on a construction by means of the Schmidt pairs of the Hankel operator [6, 8, 9] and not on transformations of the state space. The latter approach has been considered in [18, Chap. 9], where the classical result that two minimal realizations of the same input-output behavior are related by a state space transformation, the so-called *pseudo-similarity transformation*. These can be unbounded and also may not have a bounded inverse. The pseudo-similarity approach is powerful since it may be applied to general bounded well-posed linear systems.

In this article we will make use of the theory of pseudo-similarity to show that the approach in [19] to balancing and truncation of finite-dimensional linear systems can be generalized to the infinite-dimensional case. These transformations can be indeed unbounded; we will show that the operator products are however defined in a certain sense. It will also turn out that these transformations eliminate non-controllable and non-observable parts.

This article is organized as follows: Section 1 introduces the notation and basic functional analytic requisites. In Section 2 we review and develop some required results from infinite-dimensional linear systems theory. In particular, we introduce the class of systems which is considered in later parts. In Section 3 we provide a result about the generators of the well-known Kalman compression of a system, that could not be found in the existing literature and is needed in later parts. We give a definition of balanced and truncated systems in Section 4, where we furthermore show some properties of the canonical shift realizations for our system class. All the main result about balancing and normalizing transformations as well as truncation are collected in Section 5. The remaining sections are devoted to the proofs of these main theorems, except for the final Section 11, which shows the relation to the concept of pseudo-similarity if the original system is minimal.

1. Basic notation and functional analytic framework

The symbol $\mathcal{B}(X;Y)$ stands for the space of bounded linear operators from a Banach space X into the Banach space Y; we set $\mathcal{B}(X) := \mathcal{B}(X;X)$. For the evaluation of the bounded functional $f \in X' := \mathcal{B}(X;\mathbb{C})$ at $x \in X$ we write $\langle f, x\rangle_{X',X}$. To have sesquilinearity of the mapping $X' \times X \to \mathbb{C}$, $(f,x) \mapsto \langle f, x\rangle_{X',X}$, multiplication on the dual space X' is defined by

$$\langle \lambda f, x\rangle_{X',X} := \overline{\lambda}\langle f, x\rangle_{X',X}.$$

We define the inner product $\langle \cdot, \cdot\rangle_X$ in a Hilbert space X such that it is also anti-linear in the first and linear in the second component. The Riesz isomorphism $x \mapsto \langle x, \cdot\rangle_X$, which is often used to identify X with its own dual X', becomes linear in this way. Furthermore, we use $\mathrm{id}_X, \pi_Z \in \mathcal{B}(X)$ for the identity on X and the orthogonal projection onto the closed subspace Z of a Hilbert space X, respectively. If a normed space Z is densely and continuously embedded into the Banach space X, we write $Z \hookrightarrow X$.

The domain of an operator T is denoted by $\operatorname{dom} T$, the restriction of T to a subspace $Z \subset \operatorname{dom} T$ by $T|_Z$; the part of $T : \operatorname{dom} T \subset X \to Z$ in Z is defined to be the restriction of T to the domain $\{z \in Z \cap \operatorname{dom} T : Tz \in Z\}$. For the closure of an operator T we use the symbol $\overline{T}$. The kernel and range of a linear operator are denoted by $\ker T$ and $\operatorname{ran} T$, respectively. The resolvent set of T is $\rho(T) \subset \mathbb{C}$.

Definition 1.1 (Strongly continuous semigroup, generator). An operator-valued function $\mathfrak{A} : \overline{\mathbb{R}_+} \to \mathcal{B}(X)$ is called a *strongly continuous semigroup*, if $\mathfrak{A}(0) = \mathrm{id}_X$, $\mathfrak{A}(t+s) = \mathfrak{A}(t) \cdot \mathfrak{A}(s)$ for all $t, s \in \overline{\mathbb{R}_+}$, and

$$\lim_{t\to 0, t>0} \mathfrak{A}(t)x = x \quad \text{for all } x \in X.$$

A strongly continuous semigroup is called *bounded*, if there exists some $M \in \mathbb{R}_+$ such that $\|\mathfrak{A}(t)\|_{\mathcal{B}(X)} \leqslant M$ for all $t \in \mathbb{R}_+$. The operator $A : \operatorname{dom} A \subset X \to X$ defined by

$$\operatorname{dom} A = \left\{ x \in X \,\middle|\, \lim_{t\downarrow 0} \tfrac{1}{t}(\mathfrak{A}(t)x - x) \in X \right\}, \quad Ax = \lim_{t\downarrow 0} \tfrac{1}{t}(\mathfrak{A}(t)x - x),$$

is called the *generator* of the semigroup $\mathfrak{A}(\cdot)$.

The domains of A and its adjoint are known to be dense in X; A is a closed operator [22, Cor. 2.1.8 & Prop. 2.3.1 & Prop. 2.8.1]. For a strongly continuous semigroup $t \mapsto \mathfrak{A}(t)$ and appropriate operators T and T^+ we abbreviate the semigroup $t \mapsto T\mathfrak{A}(t)T^+$ by $T\mathfrak{A}T^+$. A linear space Z is called a core for the generator A of $\mathfrak{A}$, if it dense in $\operatorname{dom} A$ with respect to the graph norm of A. By [5, Proposition II.1.7] Z is already a core for A if it is dense in $\operatorname{dom} A$ with respect to the norm of X and $\mathfrak{A}$-invariant.

Lemma 1.1. *[5, p. 60] Let X, Z be a Banach spaces with $Z \hookrightarrow X$ and A be the generator of a semigroup $\mathfrak{A}$ on X. Assume that Z is $\mathfrak{A}$ invariant and $t \mapsto \mathfrak{A}(t)|_Z$ is strongly continuous with respect to the norm of Z, then the generator of $\mathfrak{A}|_Z$ is the part of A in Z.*

Let V be a Banach space and X be a Hilbert space with $V \hookrightarrow X$. Then the dual space of V with respect to X is defined as the completion of X with respect to the norm $\|x\|_{V'} := \sup_{\|v\|_V=1} |\langle x, v\rangle_X|$. If V is reflexive than this space is indeed isomorphic to the space of bounded functionals on V [22, Proposition 2.9.2].

Lemma 1.2. *[22, Proposition 2.9.3] Let V_1, V_2, X_1 and X_2 be Hilbert spaces with $V_1 \hookrightarrow X_1$ and $V_2 \hookrightarrow X_2$. Assume $A \in \mathcal{B}(V_1; X_2)$ satisfies $A^*V_2 \subset V_1$. Then A has an extension $A_{-1} : (V_1)' \to (V_2)'$ given by $\langle A_{-1}v_1', v_2\rangle_{V_2',V_2} := \langle v_1', A^*v_2\rangle_{V_1,V_2}$ for $v_1' \in V_1'$ and $v_2 \in V_2$.*

We use the notation $\mathbb{R}_+ = (0,\infty)$, $\mathbb{R}_- := (-\infty, 0)$ and $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$. For $p \in [1,\infty]$, some interval $I \in \mathbb{R}$ and a Hilbert space X, $L^p(I;X)$ denotes the Lebesgue space of measurable functions $f : I \to X$ with the property that $\int_I \|f(t)\|_X dt < \infty$. A function $f \in L^p(\mathbb{R}_+, X)$ is said to have a Lebesgue point at 0, if the limit $\lim_{t\downarrow 0} \frac{1}{t}\int_0^t f(\tau)d\tau$ exists. The space $L^1(I,X)$ is embedded into the dual space of $L^\infty(I,X)$ via the identification of $h \in L^1(I,X)$ with the functional $x \mapsto \int_I \langle h(\tau), x(\tau)\rangle_X d\tau$. For $k \in \mathbb{N}$, the Sobolev space $W^{k,p}(I;X)$ consists of all functions whose first k distributional derivatives belong to $L^p(I;X)$. The space $W_0^{k,p}(I;X)$ consists of all functions in $W^{k,p}(I;X)$ that take the value zero on the boundary of I. $W^{-k,p}(I;X)$ is defined to be the dual space of $W_0^{k,p}(I;X)$. By extension of a function defined on $J \subset I$ to zero on $J\backslash I$, we regard $L^p(J;X)$ as a subspace of $L^p(I;X)$. For $t \in \mathbb{R}$, the left shift operators $\tau^t \in \mathcal{B}(L^2(\mathbb{R};\mathcal{U}))$, $\tau_+^t \in \mathcal{B}(L^2(\mathbb{R}_+;\mathcal{Y}))$ and $\tau_-^t \in \mathcal{B}(L^2(\mathbb{R}_-;\mathcal{U}))$ are defined by

$$\tau^t u(s) := u(s+t), \quad s \in \mathbb{R}, \qquad \tau_+^t u(s) := u(s+t), \quad s \in \mathbb{R}_+,$$

$$\tau_-^t u(s) := \begin{cases} u(s+t), & s \in (-\infty, -t), \\ 0, & s \in (-t, 0). \end{cases}$$

The Hardy space $\mathcal{H}_\infty(\mathcal{U},\mathcal{Y})$ consists of all holomorphic and bounded $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued functions defined on $\mathbb{C}_+$; this space is provided with the norm

$$\|G\|_{\mathcal{H}_\infty} = \sup_{s\in\mathbb{C}_+} \|G(s)\|_{\mathcal{H}_\infty(\mathcal{U},\mathcal{Y})}.$$

ℓ_p stands for the p-summable complex sequences.

A compact operator $T \in \mathcal{B}(X;Y)$ acting between two Hilbert spaces X and Y is known to admit a singular value decomposition

$$Tx = \sum_{n=1}^{\infty} \sigma_n \langle x, u_n\rangle_X v_n$$

for some monotonically decreasing null sequence of $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ and orthonormal systems $(u_n)_{n\in\mathbb{N}}$ in X and $(v_n)_{n\in\mathbb{N}}$ in Y [16, pp. 203]. The numbers σ_n are called *singular values* and (u_n, v_n) is called *Schmidt pair* associated to σ_n. Note that, for the sake of a better notation, we allow consecutive σ_n to be equal, i.e. we ignore the multiplicity of the singular values at this stage. If

the sequence of singular values fulfills $(\sigma_n)_{n\in\mathbb{N}} \in \ell_1$, then T is called *nuclear*. A singular value decomposition of T can also be written as

$$T = V\Sigma U^*$$

with operators $\Sigma \in \mathcal{B}(\ell_2)$, $U \in \mathcal{B}(\ell_2; X)$, $V \in \mathcal{B}(\ell_2; Y)$ defined by

$$\Sigma(x_n)_{n\in\mathbb{N}} = (\sigma_n x_n)_{n\in\mathbb{N}} \tag{1.1}$$

and

$$U(x_n)_{n\in\mathbb{N}} := \sum_{n=0}^{\infty} x_n u_n, \qquad V(x_n)_{n\in\mathbb{N}} := \sum_{n=0}^{\infty} x_n v_n. \tag{1.2}$$

Here, we have assumed that there are infinitely many singular values, or, equivalently, $\operatorname{ran} T$ is infinite-dimensional. In case of k-dimensional range, ℓ_2 is replaced by $\mathbb{C}^k$ and obvious modifications have to be made. In any case, there holds $\operatorname{ran} V = \overline{\operatorname{ran} T}$, $\operatorname{ran} U = \ker T^\perp$, $U^*U = V^*V = \mathrm{id}_{\ell_2}$, $VV^* = \pi_{\overline{\operatorname{ran} T}}$ and $UU^* = \pi_{\ker T^\perp}$ and, moreover, the restrictions $U^*|_{\overline{\operatorname{ran} T}}$ and $V^*|_{\ker T^\perp}$ are both unitary. It can be seen that Σ is injective, has dense range, and is self-adjoint, whence

$$\Sigma = V^*TU = U^*T^*V.$$

Since Σ is a bounded self-adjoint operator, $\Sigma^{\frac{1}{2}}$ has a meaning: The spaces

$$\Sigma^{\frac{1}{2}}\ell_2 := \operatorname{ran}(\Sigma^{\frac{1}{2}}), \quad \Sigma\ell_2 := \operatorname{ran}(\Sigma)$$

become Hilbert spaces with the respective scalar products

$$\langle x, y\rangle_{\Sigma^{\frac{1}{2}}} := \langle \Sigma^{-\frac{1}{2}}x, \Sigma^{-\frac{1}{2}}y\rangle_{\ell_2}, \qquad \langle x, y\rangle_{\Sigma} := \langle \Sigma^{-1}x, \Sigma^{-1}y\rangle_{\ell_2}. \tag{1.3}$$

2. The system class

We review some facts from infinite-dimensional linear systems theory which are needed in later parts. We consider systems, which can formally be written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{2.1}$$

where the input $u(\cdot)$, state $x(\cdot)$ and the output $y(\cdot)$ respectively evolve in the Hilbert spaces $\mathcal{U}$, X and $\mathcal{Y}$. In the sequel we step by step collect conditions on the operators A, B, C and D involved in (2.1). The assumptions **S1**-**S6** will mean that (2.1) constitutes a bounded well-posed linear system according to [18] and thus, a meaningful solution to the equations (2.1) can be defined. Hypotheses **H1**-**H4** are assumptions on the Hankel operator, finite-dimensionality of input and output spaces, and regularity of the system.

S1 $A : \operatorname{dom}(A) \subset X \to X$ generates a strongly continuous semigroup $\mathfrak{A}$ on X.

With the aid of A the rigged Hilbert spaces X_1 and X_{-1} are constructed as follows: Take any λ in the resolvent set $\rho(A)$ of A, then $\|\cdot\|_1 := \|(\lambda - A)\cdot\|_X$ defines a norm on $\operatorname{dom} A$, which is equivalent to the graph norm $\|\cdot\|^2_{\operatorname{dom} A} := \|\cdot\|^2_X + \|A\cdot\|^2_X$. In the other direction, the Hilbert space X_{-1} is defined by the completion of X with respect to the norm $\|\cdot\|_{-1} := \|(\lambda - A)^{-1}\cdot\|_X$ and is isometrically isomorphic to the dual space of $(\operatorname{dom} A^*, \|\cdot\|_{\operatorname{dom} A^*})$ with respect to the pivot space X [22, Proposition 2.10.2]. The operator A can, by using Lemma 1.2, be extended to an operator $A|_X : X \subset X_{-1} \to X_{-1}$ that generates the semigroup $\mathfrak{A}|_{X_{-1}}$, which is the extension of $\mathfrak{A}$ to X_{-1} [22, Sec. 2.10].

S2 $B \in \mathcal{B}(\mathcal{U}; X_{-1})$.

The assumption that B maps to a larger space than X is motivated by boundary control of partial differential equations [22, Chap. 10]. Although we identify $(\operatorname{dom} A^*)'$ with X_{-1} instead of $\operatorname{dom} A^*$ itself, the bidual space $(\operatorname{dom} A^*)''$ is again identified with $\operatorname{dom} A^*$, so that we have an adjoint operator $B' \in \mathcal{B}(\operatorname{dom} A^*; \mathcal{U})$. Since B maps to X_{-1} and $\mathfrak{A}$ extends to a strongly continuous semigroup on X_{-1}, the variation of constants formula

$$x(\tau) := \mathfrak{A}(\tau)x_0 + \int_0^\tau \mathfrak{A}(\tau - \sigma)|_{X_{-1}} Bu(\sigma)d\sigma$$

defines for each $\tau \in [0, t)$ an element of X_{-1}, given initial state $x_0 \in X$ and $u \in L^2([0, t]; \mathcal{U})$. The trajectory x is called solution of $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = x_0$.

The output operator C is allowed to map from a subspace of X, which allows for example for boundary evaluations:

S3 $C : \operatorname{dom} C \to Y$ is linear with $X_1 \subset \operatorname{dom} C \subset X$ and $C|_{X_1} \in \mathcal{B}(X_1; \mathcal{Y})$.

A further assumption on the domain of C will be made in **R1**. The operator D is simply assumed to be bounded, i.e.,

S4 $D \in \mathcal{B}(\mathcal{U}; \mathcal{Y})$.

The next assumption will entail that state and output trajectories are well-defined for any square integrable input.

S5 For all $t \in \mathbb{R}_+$, $u \in L^2([0, t]; \mathcal{U})$, $x_0 \in X$, the solutions of $\dot{x}(t) = Ax(t) + Bu(t)$ with $x(0) = x_0 \in X$ fulfill
  a) $x(\tau) \in X$ for all $\tau \in [0, t]$;
  b) $x(\tau) \in \operatorname{dom} C$ for almost all $\tau \in [0, t]$.

Assumption **S5** a) means that the state trajectory, evolving by definition in X_{-1}, effectively takes values in the state space X; **S5** b) implies that the expression $Cx(\tau)$ (and thus $y(\tau)$) is meaningful for almost all $\tau \in [0, t]$.

We further assume that the system satisfies a certain stability condition:

S6 There exists some $c > 0$ such that for all $t \in \mathbb{R}_+$, $u \in L^2([0, t]; \mathcal{U})$, $x_0 \in X$, the solutions of (2.1) with $x(0) = x_0 \in X$ fulfill

$$\|y\|_{L^2([0,t];\mathcal{Y})} + \|x(t)\|_X \leqslant c \cdot \left(\|u\|_{L^2([0,t];\mathcal{U})} + \|x_0\|_X\right).$$

The above condition on the system basically comprises four properties, namely the global boundedness of the state-to-state map (that is, the boundedness of the semigroup $\mathfrak{A}$), the global boundedness of the input-to-state map, the global boundedness of the state-to-output map and the global boundedness of the input-output map. This assumption gives rise to the following mappings being well-defined and bounded:

$$\begin{aligned} &\mathfrak{B} : L^2(\mathbb{R}_-;\mathcal{U}) \to X, && \mathfrak{C} : X \to L^2(\mathbb{R}_+;\mathcal{Y}), \\ &\quad u(\cdot) \mapsto \int_{-\infty}^{0} \mathfrak{A}(-\tau)Bu(\tau), && \quad x \mapsto C\mathfrak{A}(\cdot)x, \\ &\mathfrak{D} : L^2(\mathbb{R};\mathcal{U}) \to L^2(\mathbb{R};\mathcal{Y}), \\ &\quad u(\cdot) \mapsto C\int_{-\infty}^{\cdot} \mathfrak{A}(\cdot-\tau)Bu(\tau)d\tau + Du(\cdot). \end{aligned} \tag{2.2}$$

The *controllability map* $\mathfrak{B}$ is the operator that maps past input to state at zero time; the *observability map* $\mathfrak{C}$ applied to $x \in X$ consists of the output trajectory $y(\cdot)$ of the system with zero input initialized with x; the *input-output map* $\mathfrak{D}$ maps the input $u(\cdot) \in L^2(\mathbb{R};\mathcal{U})$ of the system to the corresponding output $y(\cdot)$. The latter one has the following two properties, namely $\mathfrak{D}$ is

(i) *time-invariant*, i.e., $\tau_t\mathfrak{D} = \mathfrak{D}\tau_t$ for all $t \in \mathbb{R}$, and
(ii) *causal*, i.e. $\pi_{L^2(\mathbb{R}_-;\mathcal{Y})}\mathfrak{D}\pi_{L^2(\mathbb{R}_+;\mathcal{U})} = 0$,

The above defined mappings $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ form a well-posed linear systems according to [18, Definition 2.2.1]. The definition is recapped below.

Definition 2.1 (Well-posed linear system, realization, generator). Let $\mathcal{U}$, X and $\mathcal{Y}$ be Hilbert spaces. A bounded well-posed linear system on $(\mathcal{U}, X, \mathcal{Y})$ consists of a quadruple $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ with the following properties:

(i) $t \mapsto \mathfrak{A}(t) \in \mathcal{B}(X)$ is a bounded semigroup on X;
(ii) $\mathfrak{B} \in \mathcal{B}(L^2(\mathbb{R}_-;\mathcal{U}); X)$ satisfies $\mathfrak{A}(t)\mathfrak{B} = \mathfrak{B}\tau_-^t$ for all $t \geqslant 0$;
(iii) $\mathfrak{C} \in \mathcal{B}(X; L^2(\mathbb{R}_+;\mathcal{Y}))$ satisfies $\mathfrak{C}\mathfrak{A}(t) = \tau_+^t\mathfrak{C}$ for all $t \geqslant 0$;
(iv) $\mathfrak{D} \in \mathcal{B}(L^2(\mathbb{R};\mathcal{U}); L^2(\mathbb{R};\mathcal{Y}))$ is continuous, causal, time-invariant and it satisfies $\pi_{L^2(\mathbb{R}_+;\mathcal{Y})}\mathfrak{D}|_{L^2(\mathbb{R}_-;\mathcal{U})} = \mathfrak{C}\mathfrak{B}$.

X is called the state space of the system, and since $\mathcal{U}$ and $\mathcal{Y}$ are fixed in this paper, we just speak of a system *on* X. Any bounded well-posed system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ satisfying these conditions is called a realization of its input-output map $\mathfrak{D}$. Furthermore, if a well-posed linear system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is defined via (2.2) with a quadruple (A, B, C, D) satisfying **S1-S6**, then we call (A, B, C, D) the *generators* of $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$.

A bounded well-posed linear system is called *observable*, if $\ker \mathfrak{C} = \{0\}$, *controllable*, if $\operatorname{ran} \mathfrak{B}$ is dense in X, and *minimal*, if it is both, controllable and observable.

Remark 2.1 (Well-posed linear systems). Definition 2.1 actually covers a larger class than the one fulfill **S1-S6**. More precisely, the class of systems that can be described by **S1-S6** is called *compatible, bounded L^2-well posed systems* in [18].

Definition 2.2 (Hankel operator, Gramians). For a bounded well-posed linear system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ on $(\mathcal{U}, X, \mathcal{Y})$,

(i) $\mathfrak{H} = \mathfrak{C}\mathfrak{B} \in \mathcal{B}(L^2(\mathbb{R}_-;\mathcal{U}); L^2(\mathbb{R}_+;\mathcal{Y}))$ is called *Hankel operator*,
(ii) $P = \mathfrak{B}\mathfrak{B}^* \in \mathcal{B}(X)$ is called *controllability Gramian*,
(iii) $Q = \mathfrak{C}^*\mathfrak{C} \in \mathcal{B}(X)$ is called *observability Gramian*.

While **S1**–**S6** will be permanently presumed throughout this work, the following assumptions will be optional.

H1 $\mathcal{U}$ and $\mathcal{Y}$ are finite-dimensional.

The following are compactness assumptions on the Hankel operator of varying strength.

H2 The Hankel operator is nuclear.
H3 The Hankel operator has a special representation: Namely, there exists some $h \in L^1(\mathbb{R}_+; \mathcal{B}(\mathcal{U};\mathcal{Y}))$, such that for all $u \in L^2(\mathbb{R}_-;\mathcal{U})$ holds

$$(\mathfrak{H}u)(t) = \int_{-\infty}^{0} h(t-\tau)u(\tau)d\tau, \text{ for almost all } t \in \mathbb{R}_+. \tag{2.3}$$

H4 The Hankel operator is compact.

Remark 2.2 (Systems with nuclear Hankel operator). a) The Hankel operator as used here is related to the Hankel operator used in [8] and [6] via multiplication from the left with the reflection operator

$$\Gamma : L^2(\mathbb{R}_-;\mathcal{Y}) \to L^2(\mathbb{R}_+;\mathcal{Y}) \qquad y(\cdot) \mapsto y(-\cdot). \tag{2.4}$$

b) Given **H1**, the implications **H2**⇒**H3**⇒**H4** hold. The first implication has been proven in [8, Corollary 5.1.18.], the second one in [6, Appendix 1, p.895].
c) Further characterizations of nuclearity of Hankel operators can be found in [4].
d) Compactness of the Hankel operator gives rise to the existence of a singular value decomposition of the Hankel operator, that is,

$$\mathfrak{H} = \widetilde{V}\Sigma\widetilde{U}^*,$$

with diagonal operator $\Sigma \in \mathcal{B}(\ell_2)$ as in (1.1). The elements of the strictly decreasing sequence $(\sigma_n)_{n\in\mathbb{N}}$ are called *Hankel singular values*.

Definition 2.3 (Regular linear system). A well-posed system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is said to be *regular*, if there is an operator $D \in \mathcal{B}(\mathcal{U};\mathcal{Y})$, such that for all $v \in \mathcal{U}$ holds

$$\lim_{t\to 0} \frac{1}{t}\int_0^t \mathfrak{D}(\chi_{[0,t]}v)(\tau)d\tau = Dv, \tag{2.5}$$

where $\chi_{[0,t]} : \mathbb{R} \to \mathbb{R}$ is the characteristic function on $[0,t]$. The operator D with the above property is called *feedthrough operator*.

The final assumption is again concerned with the output operator C and closely related to regularity.

R1 The system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ generated by (A, B, C, D) is regular and for all $x \in \operatorname{dom} C$ holds

$$Cx = C_L x := \lim_{t \to 0} \frac{1}{t} \int_0^t (\mathfrak{C}x)(\tau) d\tau. \tag{2.6}$$

Moreover, $\operatorname{dom} C$ consists of all $x \in X$ for which the limit in (2.6) exists.

Definition 2.4. The operator C_L from equation (2.6) with its natural domain is called *Lebesgue extension* of $C_L|_{\operatorname{dom} A}$.

Given an arbitrary regular well-posed system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$, it is possible to assign a unique generator (A, B, C, D) that satisfies **S1**-**S6** and **R1** to it: Obviously, A has to generate the semigroup $\mathfrak{A}$. Furthermore, according to [18, Thm 4.2.1], B can be defined to be

$$Bu := (\lambda - A|_X)\mathfrak{B}(e_\lambda u), \qquad u \in \mathcal{U} \tag{2.7}$$

where $e_\lambda \in L^2(\mathbb{R}_-; \mathbb{C})$ is the function $t \mapsto e^{\lambda t}$ for some $\lambda \in \rho(A)$. C and D are defined via (2.6) and (2.5), where the domain of C is by definition the set on which which the limit in (2.6) exists.

Remark 2.3 (Regular linear systems, transfer functions). a) In [18], the mapping $C_L|_{\operatorname{dom} A}$ is denoted by C and said to be the generator of $\mathfrak{C}$. In contrast to this, we have defined the generator C to be the Lebesgue extension of $C|_{\operatorname{dom} A}$ here.

b) There is some redundancy in **R1**: Assume that **S1**–**S6** hold. Then by [21, Theorem 5.8], **R1** already implies regularity of the generated system and also that D fulfills (2.5). On the other hand, if the system is regular, C and D can always be redefined by (2.5) and (2.6) to make (2.6) hold.

c) Regularity is implied by **S1**–**S6** and **H2**: Using

$$(\mathfrak{D}u)(t) = \int_{-\infty}^t h(t-\tau)u(\tau)ds + Du(t), \text{ for almost all } t \in \mathbb{R},$$

and $h \in L^1(\mathbb{R}_+; \mathcal{B}(\mathcal{U}; \mathcal{Y}))$, we see that (2.5) holds for all $v \in \mathcal{U}$. In particular, if **S1**–**S6** and **H2** are fulfilled, **R1** can be assumed without loss of generality. Relation (c) in particular implies that the input-output map $\mathfrak{D}$ is uniquely determined by $\mathfrak{H}$ and D.

d) It follows from [21, Theorem 5.8] that regularity implies $(\operatorname{ran}(sI - A)^{-1}B) \subset \operatorname{dom} C$ for all $s \in \mathbb{C}_+$. This gives rise to the existence of the *transfer function* $G : \mathbb{C}_+ \to \mathcal{B}(\mathcal{U}; \mathcal{Y})$, which is defined by

$$G(s) = C(sI - A)^{-1}B + D.$$

There holds $G \in \mathcal{H}_\infty(\mathcal{U}; \mathcal{Y})$ [20] and

$$\|G\|_{\mathcal{H}_\infty} = \|\mathfrak{D}\|_{\mathcal{B}(L^2(\mathbb{R};\mathcal{U}), L^2(\mathbb{R};\mathcal{Y}))}.$$

We will very frequently make use of the following basic assertion on similarity of well-posed linear systems, which can be found in [18, Example 2.3.7].

Lemma 2.1. *Given a well-posed linear system* $(\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{C}_1, \mathfrak{D}_1)$ *on* $(\mathcal{U}, X, \mathcal{Y})$*, a further Hilbert space* Z *and a boundedly invertible operator* $T \in \mathcal{B}(X; Z)$*. Then* $\mathfrak{A}_2 : \mathbb{R}_+ \to \mathcal{B}(Z)$*,* $t \mapsto T\mathfrak{A}_1(t)T^{-1}$*,* $\mathfrak{B}_2 := T\mathfrak{B}_1$*,* $\mathfrak{C}_2 := \mathfrak{C}_1 T^{-1}$ *and* $\mathfrak{D}_2 := \mathfrak{D}_1$ *constitute a well-posed linear system on* $(\mathcal{U}, Z, \mathcal{Y})$*. If the system is regular, the generators of this system are given by* (A_2, B_2, C_2, D) *with* $\operatorname{dom} A_2 = T \operatorname{dom} A_1$*,* $\operatorname{dom} C_2 = T \operatorname{dom} C_1$ *and*

$$A_2 = TA_1T^{-1}, \quad B_2 = T|_{(\operatorname{dom} A_1^*)'} B_1, \quad C_2 = C_1T^{-1}, \quad D_2 = D_1.$$

Here, $T|_{(\operatorname{dom} A_1^*)'}$ *is the unique extension of* T *to an operator from* $(\operatorname{dom} A_1^*)'$ *to* $(\operatorname{dom} A_2^*)'$*.*

A classical result in finite-dimensional linear systems theory is that two minimal systems with equal input-output map are similar. The concept of pseudo-similarity generalizes this to well-posed linear systems.

Definition 2.5. Two well-posed linear systems $(\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{C}_1, \mathfrak{D})$ and $(\mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{C}_2, \mathfrak{D}_2)$ on $(\mathcal{U}, X, \mathcal{Y})$, respectively $(\mathcal{U}, Z, \mathcal{Y})$ are *pseudo-similar*, if $\mathfrak{D}_1 = \mathfrak{D}_2$, and there exists a closed, densely defined injective linear operator $T : \operatorname{dom} T \subset X \to \operatorname{ran} T \subset Z$ with the following properties: $\operatorname{ran} \mathfrak{B}_1 \subset \operatorname{dom} T$, $\operatorname{ran} \mathfrak{B}_2 \subset \operatorname{ran} T$, $\operatorname{dom} T$ is $\mathfrak{A}_1$-invariant, $\operatorname{ran} T$ is $\mathfrak{A}_2$-invariant and

$$\begin{aligned} \mathfrak{A}_2(t)Tx_1 &= T\mathfrak{A}_1(t)x_1, && \forall x_1 \in \operatorname{dom} T, t \in \mathbb{R}_+, \\ \mathfrak{B}_2 u &= T\mathfrak{B}_1 u, && \forall u \in L^2(\mathbb{R}_-; \mathcal{U}), \\ \mathfrak{C}_2 Tx_1 &= \mathfrak{C}_1 x_1, && \forall x_1 \in \operatorname{dom} T. \end{aligned}$$

If T and T^{-1} are both bounded (unitary), then $(\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{C}_1, \mathfrak{D}_1)$ and $(\mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{C}_2, \mathfrak{D}_1)$ are called (unitarily) similar.

3. Kalman compression

The principle of restricting a system to its approximately controllable and observable subspaces is known for abstract linear systems [18, Corollary 9.1.10]. In addition to this, we need to know what the generators of such a restriction are.

Theorem 3.1 (Kalman compression). *Let* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ *be a regular, bounded well-posed linear system on* $(\mathcal{U}, X, \mathcal{Y})$*. With the definitions*

$$M := \pi_{(\ker \mathfrak{C})^\perp} \operatorname{ran} \mathfrak{B} = \operatorname{ran}(\pi_{(\ker S^*)^\perp} R)$$
$$\widetilde{\mathfrak{A}} := \pi_{(\ker \mathfrak{C})^\perp} \mathfrak{A}|_{\overline{M}}, \qquad \widetilde{\mathfrak{B}} := \pi_{(\ker \mathfrak{C})^\perp} \mathfrak{B} \qquad \widetilde{\mathfrak{C}} := \mathfrak{C}|_{\overline{M}},$$

the quadruple $(\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \widetilde{\mathfrak{C}}, \mathfrak{D})$ *is a regular and minimal, bounded well-posed linear system on* $(\mathcal{U}, \overline{M}, \mathcal{Y})$*. The generator* $\widetilde{A}$ *of* $\widetilde{\mathfrak{A}}$ *is given by*

$$\operatorname{dom} \widetilde{A} = \overline{M} \cap \pi_{(\ker \mathfrak{C})^\perp} \operatorname{dom} A, \qquad \widetilde{A}x = \pi_{(\ker \mathbb{C})^\perp} Az$$

for $z \in \operatorname{dom} A$ *such that* $x = \pi_{(\ker \mathfrak{C})^\perp} z \in \operatorname{dom} \widetilde{A}$. *The domain of the adjoint operator* $\widetilde{A}^*$ *is* $\pi_{\overline{M}}(\operatorname{dom} A^* \cap (\ker \mathfrak{C})^\perp)$. *The generator* $\widetilde{B}$ *is given by*

$$\langle \widetilde{B}u, x\rangle_{(\operatorname{dom} \widetilde{A}^*)', \operatorname{dom} \widetilde{A}^*} = \langle Bu, z\rangle_{(\operatorname{dom} A^*)', \operatorname{dom} A^*}$$

for $z \in (\ker \mathfrak{C})^\perp \cap \operatorname{dom} A^*$ *such that* $\pi_{\overline{M}} z = x$. *And* $\widetilde{C}x = Cx$ *for all* $x \in \operatorname{dom} \widetilde{C} = \operatorname{dom} C \cap \overline{M}$.

We will divide the proof into two lemmas.

Lemma 3.2. *Under the assumptions of Lemma* 3.1 *define* $Z := \ker \mathfrak{C}$. *Then*

$$\left(\widetilde{\mathfrak{A}}, \quad \widetilde{\mathfrak{B}}, \quad \widetilde{\mathfrak{C}}, \quad \mathfrak{D}\right) := \left(\pi_{Z^\perp}\mathfrak{A}|_{Z^\perp}, \quad \pi_{Z^\perp}\mathfrak{B}, \quad \mathfrak{C}|_{Z^\perp}, \quad \mathfrak{D}\right)$$

is a regular, bounded well-posed linear system on $(\mathcal{U}, Z^\perp, \mathcal{Y})$. *Its generators are* $(\widetilde{A}, \widetilde{B}, \widetilde{C}, D)$, *where*

$$\widetilde{A}z = \pi_{Z^\perp} Ax \quad \text{for any } x \in \operatorname{dom} A \text{ with } \pi_{Z^\perp} x = z, \quad \operatorname{dom} \widetilde{A} = \pi_{Z^\perp} \operatorname{dom} A,$$

$$\widetilde{A}^* z = A^* z \quad \forall z \in \operatorname{dom} \widetilde{A}^* = Z^\perp \cap \operatorname{dom} A^*$$

$$\langle \widetilde{B}u, z\rangle_{(\operatorname{dom} \widetilde{A}^*)', \operatorname{dom} \widetilde{A}^*} = \langle Bu, z\rangle_{(\operatorname{dom} A^*)', \operatorname{dom} A^*} \quad \forall z \in \operatorname{dom} \widetilde{A}^*$$

$$\widetilde{C}z = Cx, \quad \text{for any } x \in \operatorname{dom} C \text{ with } \pi_{Z^\perp} x = z \operatorname{dom} \widetilde{C} = \pi_{Z^\perp} \operatorname{dom} C.$$

Proof. The part about the system operators is shown in [18, Corollary 9.1.10]. Note that Z is an $\mathfrak{A}$-invariant, closed subspace. The generator $\widetilde{A}$ of the quotient semigroup $\widetilde{\mathfrak{A}}$ can be found in [5, Section 2.2.4]. Since we are in a Hilbert space setting, the adjoint semigroups $\mathfrak{A}^*$ and $\widetilde{\mathfrak{A}}^*$ are again strongly continuous [18, Theorem 3.5.6] Note that the $\mathfrak{A}$-invariance of Z implies the invariance of $Z^\perp$ under $\mathfrak{A}^*$ and therefore a quick calculation shows $\widetilde{\mathfrak{A}}^* = \mathfrak{A}^*|_{Z^\perp}$. Thus, the generators $\widetilde{A}^*$ and $A^*|_{Z^\perp}$ must coincide and the extension $\widetilde{A}|_{Z^\perp} : Z^\perp \subset (\operatorname{dom} \widetilde{A}^*)' \to (\operatorname{dom} \widetilde{A}^*)'$ reads

$$\langle \widetilde{A}|_{Z^\perp} z, y\rangle_{(\operatorname{dom} \widetilde{A}^*)', \operatorname{dom} \widetilde{A}^*} = \langle z, A^* y\rangle_X \quad \forall y \in \operatorname{dom} \widetilde{A}^*.$$

We use this to calculate $\widetilde{B}$ via (2.7). Using that $Z^\perp$ is A^*-invariant we obtain for all $z \in \operatorname{dom} \widetilde{A}^*$

$$\begin{aligned}\langle \widetilde{B}u, z\rangle_{(\operatorname{dom} \widetilde{A}^*)', \operatorname{dom} \widetilde{A}^*} &= \langle \widetilde{\mathfrak{B}} e_\lambda u, (\overline{\lambda} - \widetilde{A}^*) z\rangle_{Z^\perp} = \langle \pi_{Z^\perp} \mathfrak{B} e_\lambda u, \overline{\lambda} z - A^* z\rangle_X \\ &= \langle \mathfrak{B} e_\lambda u, \overline{\lambda} z - A^* z\rangle_X = \langle (\lambda - A|_X) \mathfrak{B} e_\lambda u, z\rangle_X \\ &= \langle Bu, z\rangle_{(\operatorname{dom} A^*)', \operatorname{dom} A^*}.\end{aligned}$$

Now we turn to $\widetilde{C}$. First assume that $x \in \operatorname{dom} C \cap Z^\perp$, then

$$\frac{1}{t}\int_0^t (\widetilde{\mathfrak{C}}x)(s)ds = \frac{1}{t}\int_0^t (\mathfrak{C}x)(s)ds \xrightarrow{t\downarrow 0} Cx,$$

and in particular $x \in \operatorname{dom} \widetilde{C}$. Conversely, let $x \in \operatorname{dom} \widetilde{C}$. Then, $x \in Z^\perp$ by definition, and the equality

$$\frac{1}{t}\int_0^t (\mathfrak{C}x)(s)ds = \frac{1}{t}\int_0^t (\widetilde{\mathfrak{C}}x)(s)ds \xrightarrow{t\downarrow 0} \widetilde{C}x,$$

shows $x \in \operatorname{dom} C$. □

Lemma 3.3. *Under the assumptions of Lemma* 3.1*, define* $Z := \overline{\operatorname{ran} \mathfrak{B}}$*. Then*

$$\begin{pmatrix} \widetilde{\mathfrak{A}}, & \widetilde{\mathfrak{B}}, & \widetilde{\mathfrak{C}}, & \mathfrak{D} \end{pmatrix} := \begin{pmatrix} \mathfrak{A}|_Z, & \mathfrak{B}, & \mathfrak{C}|_Z, & \mathfrak{D} \end{pmatrix}$$

is a regular, bounded well-posed linear system on $(\mathcal{U}, Z, \mathcal{Y})$*. Its generators are* $(\widetilde{A}, \widetilde{B}, \widetilde{C}, D)$*, where*

$\widetilde{A}z = Az, \quad \operatorname{dom} \widetilde{A} = Z \cap \operatorname{dom} A,$

$\widetilde{A}^* z = \pi_Z A^* x$ *for any* $x \in \operatorname{dom} A^*$ *with* $\pi_Z x = z, \quad \operatorname{dom} \widetilde{A}^* = \pi_Z \operatorname{dom} A^*$

$\langle \widetilde{B}u, z \rangle_{(\operatorname{dom} \widetilde{A}^*)', \operatorname{dom} \widetilde{A}^*} = \langle Bu, x \rangle_{(\operatorname{dom} A^*)', \operatorname{dom} A^*}$ *for any* $x \in \operatorname{dom} A^*$ *with* $\pi_Z x = z,$

$\widetilde{C}z = Cx, \quad \operatorname{dom} \widetilde{C} = Z \cap \operatorname{dom} C.$

Proof. The part about the system operators is easy and well-known. Since we are in a Hilbert space setting, the adjoint semigroups $\mathfrak{A}^*$ and $\widetilde{\mathfrak{A}}^*$ are again strongly continuous [18, Theorem 3.5.6], and $\widetilde{A}^*$ generates the latter. A short calculation shows that $\widetilde{\mathfrak{A}}^* = \pi_Z \mathfrak{A}^*|_Z$. So $\widetilde{A}^*$ can alternatively be characterized as the quotient generator of the quotient semigroup, which has by [5, Section 2.2.4] the asserted representation. Therefore, the extension $\widetilde{A}|_Z : Z \subset (\operatorname{dom} \widetilde{A}^*)' \to (\operatorname{dom} \widetilde{A}^*)'$ is for all $z \in Z$ given by

$$\langle \widetilde{A}|_Z z, y \rangle_{(\operatorname{dom} \widetilde{A}^*)', \operatorname{dom} \widetilde{A}^*} = \langle z, \pi_Z A^* x \rangle_X \quad \forall x \in \operatorname{dom} A^* \text{ with } \pi_Z x = y.$$

We use this to resolve for $u \in U$ the expression $\widetilde{B}u = (\lambda - \widetilde{A})\widetilde{\mathfrak{B}} e_\lambda u$: We take an arbitrary $z \in \operatorname{dom} \widetilde{A}^*$ and some $x \in \operatorname{dom} A^*$ with $\pi_Z x = z$.

$$\begin{aligned} \langle \widetilde{B}u, z \rangle_{(\operatorname{dom} \widetilde{A}^*)', \operatorname{dom} \widetilde{A}^*} &= \langle \widetilde{\mathfrak{B}} e_\lambda u, (\overline{\lambda} - \widetilde{A}^*) z \rangle_Z = \langle \mathfrak{B} e_\lambda u, \overline{\lambda} z - \pi_Z A^* x \rangle_X \\ &= \langle \mathfrak{B} e_\lambda u, \overline{\lambda} x - A^* x \rangle_X = \langle (\lambda - A|_X) \mathfrak{B} e_\lambda u, x \rangle_X \\ &= \langle Bu, x \rangle_{(\operatorname{dom} A^*)', \operatorname{dom} A^*}. \end{aligned}$$

The part about $\widetilde{C}$ is a direct consequence of the definition (2.6), including the domain. □

Proof of Theorem 3.1. The theorem follows by applying first Lemma 3.2 and then Lemma 3.3 with $Z = \overline{\operatorname{ran} \pi_{(\ker \mathfrak{C})^\perp} \mathfrak{B}} = \overline{M}$. The only thing that remains to be proven is that the projections $\pi_{\overline{M}}$ and $\pi_{(\ker \mathfrak{C})^\perp}$ coincide on $\operatorname{ran} \mathfrak{B}$, or, in other words

$$\pi_{\overline{M}} \mathfrak{B} u = \pi_{(\ker \mathfrak{C})^\perp} \mathfrak{B} u \quad \forall u \in L^2(\mathbb{R}_-; \mathcal{U})$$

Indeed, from $\overline{M} = \overline{\operatorname{ran}(\pi_{(\ker \mathfrak{C})^\perp} \mathfrak{B})} \subset \overline{(\ker \mathfrak{C})^\perp} = (\ker \mathfrak{C})^\perp$ we deduce for arbitrary u

$$\begin{aligned} & \mathfrak{B} u = \pi_{\overline{M}} \mathfrak{B} u + \pi_{\overline{M}^\perp} \mathfrak{B} u = \pi_{(\ker \mathfrak{C})^\perp} \mathfrak{B} u + \pi_{\ker \mathfrak{C}} \mathfrak{B} u \\ \Rightarrow \quad & \pi_{\overline{M}} \mathfrak{B} u - \pi_{(\ker \mathfrak{C})^\perp} \mathfrak{B} u = \pi_{\ker \mathfrak{C}} \mathfrak{B} u - \pi_{\overline{M}^\perp} \mathfrak{B} u \in \overline{M} \cap \overline{M}^\perp = \{0\}. \end{aligned}$$

□

Remark 3.1 (Kalman compression). Theorem 3.1 still holds for the wider class of (possibly unstable) L^p-well-posed linear systems.

4. Normalized, balanced and truncated realizations

We will now introduce balanced realizations of well-posed linear systems. As in the finite-dimensional case, this involves both Gramians being equal to some diagonal operator Σ.

Definition 4.1 (Normalized and balanced systems). A bounded well-posed linear system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ on $(\mathcal{U}, X, \mathcal{Y})$ is called *input normalized* if $P := \mathfrak{B}\mathfrak{B}^* = \mathrm{id}_X$, and *output normalized* if $Q := \mathfrak{C}^*\mathfrak{C} = \mathrm{id}_X$. The system is called *balanced* if $X = \ell_2$ and there exists some positive and strictly decreasing sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that the Gramians P and Q are both equal to the diagonal operator Σ defined in (1.1). In other words

$$P = \mathfrak{B}\mathfrak{B}^* = Q = \mathfrak{C}^*\mathfrak{C} = \Sigma.$$

Remark 4.1.
a) The sequence $(\sigma_n)_{n\in\mathbb{N}}$ in Definition 4.1 consists indeed of the Hankel singular values of the system.
b) Balanced realizations are minimal.
c) Our definition of a balanced system is stronger than the one in [18, Sec. 5.5] for general well-posed linear systems. There, a system is already called balanced if both Gramians are equal. The latter property is called *parbalanced* in [15], and does not require the Hankel operator to possess a singular value decomposition. Our definition is motivated by the original one for the finite-dimensional case [11], where Σ is assumed to be a diagonal matrix with decreasing diagonal elements.

We give a definition of the well-known shift realizations. For our purposes we will in particular need the realizations on the range of the Hankel operator.

Lemma 4.1. *Consider a bounded well-posed linear system* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ *on* $(\mathcal{U}, X, \mathcal{Y})$.

(i) Let $Z := (\ker \mathfrak{H})^\perp \subset L^2(\mathbb{R}_-; \mathcal{U})$. *By the* exactly controllable shift realization *of* $\mathfrak{D}$ *on* Z, *we mean the system*

$$\left(\pi_Z \tau_-|_Z, \quad \pi_Z, \quad \mathfrak{H}|_Z, \quad \mathfrak{D}\right) \tag{4.1}$$

which is bounded and well-posed with state space Z. If, in addition, **H3** *and* **R1** *hold, the generators of this system are given by*

$$A : \operatorname{dom} A \subset Z \to Z, \quad \operatorname{dom} A = \pi_Z W_0^{1,2}(\mathbb{R}_-;\mathcal{U}),$$
$$(Az)(\xi) = \tfrac{d}{d\xi}x(\xi), \qquad \textit{for any } x \in W_0^{1,2}(\mathbb{R}_-;\mathcal{U}) \textit{ with } \pi_Z x = z.$$
$$B : \mathcal{U} \to Z_{-1},\ u \mapsto u\langle \delta(0), \cdot\rangle,$$
$$C : \operatorname{dom} C \to \mathcal{Y}, \quad \operatorname{dom} C = \left\{ x \in L^2(\mathbb{R}_-;\mathcal{U}) \;\middle|\; \begin{array}{l} \int_{-\infty}^0 h(\cdot - \tau)x(\tau)d\tau \textit{ has} \\ \textit{a Lebesgue point at } 0. \end{array} \right\},$$
$$Cx = \lim_{t\downarrow 0} \frac{1}{t}\int_0^t \int_{-\infty}^0 h(\xi - \tau)x(\tau)d\tau d\xi.$$

(ii) Let $Z := \overline{\operatorname{ran}\mathfrak{H}} \subset L^2(\mathbb{R}_+;\mathcal{Y})$. By the exactly observable shift realization *of $\mathfrak{D}$ on Z, we mean the system*

$$\big(\tau_+|_Z, \quad \mathfrak{H}, \quad \mathrm{id}_Z, \quad \mathfrak{D}\big) \tag{4.2}$$

which is bounded and well-posed with state space Z. If, in addition, **H3** *and* **R1** *hold, then* (4.2) *is generated by (A, B, C, D) with D as in* (2.5) *and*

$$A : \operatorname{dom} A \subset Z \to Z, \quad \operatorname{dom} A = W^{1,2}(\mathbb{R}_+;\mathcal{Y}) \cap Z,$$
$$(Ax)(\xi) = \tfrac{d}{d\xi}x(\xi),$$
$$B : \mathcal{U} \to Z_{-1}, \quad \langle Bu, z\rangle = \int_0^\infty h(\xi)ux(\xi)d\xi \qquad \begin{array}{l} \textit{for any } x \in W_0^{1,2}(\mathbb{R}_-;\mathcal{U}) \\ \textit{with } \pi_Z x = z, \end{array}$$
$$C : \operatorname{dom} C \to \mathcal{Y}, \quad \operatorname{dom} C = \{x \in Z : x \textit{ has a Lebesgue point at } 0\},$$
$$Cx = \lim_{t\to 0} \frac{1}{t}\int_0^t x(\tau)d\tau.$$

For the adjoint of the generator we have

$$A^* : \operatorname{dom} A^* \subset Z \to Z, \quad \operatorname{dom} A^* = \pi_Z W_0^{1,2}(\mathbb{R}_+;\mathcal{Y}),$$
$$(A^*z)(\xi) = -\pi_Z \tfrac{d}{d\xi}x(\xi) \textit{ for any } x \in W_0^{1,2}(\mathbb{R}_-;\mathcal{U}) \textit{ with } \pi_Z x = z.$$

Proof. The well-posedness of both systems follows from the well-posedness of the shift realizations on $L^2(\mathbb{R}_+;\mathcal{Y})$, respectively $L^2(\mathbb{R}_-;\mathcal{U})$, described in [18, Example 2.6.5] and our Lemmas 3.2 and 3.3. The semigroup generators are a result of Lemma 3.3 applied to the generator of the shift realization on $L^2(\mathbb{R}_-;\mathcal{U})$, respectively $L^2(\mathbb{R}_+;\mathcal{Y})$, which can be found in [18, Example 3.2.3(ii)]. The operator A^* is obtained in the same manner, since it is the generator of the right shift, which is adjoint to the left shift. We only discuss the generators for (ii): The verification of the operator C is straightforward, so we only calculate B via (2.7): Let $A|_{L^2(\mathbb{R}_+;\mathcal{Y})}$ be the distributional derivative on $L^2(\mathbb{R}_+;\mathcal{Y})$ and $\lambda \in \mathbb{C}_+$. Then for all $z \in \operatorname{dom} A^*$ we take an arbitrary

$x \in W_0^{1,2}(\mathbb{R}_+;\mathcal{Y})$ with $\pi_Z x = z$ and

$$\begin{aligned}\langle Bu, y\rangle_{(\operatorname{dom} A^*)',\operatorname{dom} A^*} &= \langle (\lambda - A)|_{L^2(\mathbb{R}_+;\mathcal{Y})}\mathfrak{H}e_\lambda u, z\rangle_{(\operatorname{dom} A^*)',\operatorname{dom} A^*}\\ &= \langle \mathfrak{H}e_\lambda u, (\overline{\lambda} - A^*)z\rangle_{(\operatorname{dom} A^*)',\operatorname{dom} A^*}\\ &= \langle \mathfrak{H}e_\lambda u, \overline{\lambda}x\rangle_{L^2(\mathbb{R}_+;\mathcal{Y})} + \langle \mathfrak{H}e_\lambda u, \tfrac{d}{d\xi}x\rangle_{L^2(\mathbb{R}_+;\mathcal{Y})}\end{aligned}$$

The last term on the right hand side is

$$\begin{aligned}\left\langle \mathfrak{H}e_\lambda u, \tfrac{d}{d\xi}x\right\rangle_{L^2(\mathbb{R}_+;\mathcal{Y})} &= \int_0^\infty \left\langle \int_{-\infty}^0 h(\xi-\tau)e^{\lambda\tau}u\,d\tau, \tfrac{d}{d\xi}x(\xi)\right\rangle_{\mathcal{Y}} d\xi\\ &= \int_0^\infty \left\langle \int_{-\infty}^0 h(\xi-\tau)e^{\lambda\tau}u\,d\tau, \tfrac{d}{d\xi}x(\xi)\right\rangle_{\mathcal{Y}} d\xi\\ &= -\int_0^\infty \left\langle \int_{-\infty}^\xi h(\tau)e^{\lambda(\xi-\tau)}u\,d\tau, \tfrac{d}{d\xi}x(\xi)\right\rangle_{\mathcal{Y}} d\xi.\end{aligned}$$

Since the inner integral is differentiable with respect to ξ this becomes

$$\begin{aligned}&= \int_0^\infty \left\langle \tfrac{d}{d\xi}\int_{-\infty}^\xi h(\tau)e^{\lambda(\xi-\tau)}u\,d\tau, x(\xi)\right\rangle_{\mathcal{Y}} d\xi\\ &= \int_0^\infty \left\langle h(\xi)u + \int_{-\infty}^\xi h(\tau)\lambda e^{\lambda(\xi-\tau)}u\,d\tau, x(\xi)\right\rangle_{\mathcal{Y}} d\xi\\ &= \int_0^\infty \left\langle h(\xi)u - \lambda\int_{-\infty}^0 h(\xi-\tau)e^{\lambda(\tau)}u\,d\tau, x(\xi)\right\rangle_{\mathcal{Y}} d\xi.\end{aligned}$$

Plugging this into the original equation yields the asserted expression for Bu. □

Remark 4.2. The exactly controllable shift realization on $(\ker\mathfrak{H})^\perp$ is observable and the exactly observable shift realization on $\overline{\operatorname{ran}\mathfrak{H}}$ is controllable.

The following definition of a balanced truncation is taken from [8]. One of our aims is to show that this system is indeed obtained from a given state space-system by first balancing an then truncating it in an appropriate sense. That is, the balanced truncation does deserve its name.

Definition 4.2. Let a system (2.1) fulfilling **S1**-**S6**, **R1**, **H1** and **H3** be given. Let $(\sigma_n)_{n\in\mathbb{N}}$ be the sequence of singular values with corresponding Schmidt pairs $(\widetilde{v}_j, \widetilde{u}_j)$ of the Hankel operator $\mathfrak{H}$. Choose r such that $\sigma_{r+1} \neq \sigma_r$.

(i) The *r-th order truncated balanced system* is defined to be

$$\begin{aligned}\dot{x}_r(t) &= A_r x_r(t) + B_r u(t),\\ y_r(t) &= C_r x_r(t) + Du(t),\end{aligned} \tag{4.3}$$

where the matrices

$$A_r = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{bmatrix} \in \mathbb{C}^{r,r}, \quad B_r = \begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix} \in \mathcal{B}(\mathcal{U}, \mathbb{C}^r),$$
$$C_r = \begin{bmatrix} c_1 & \cdots & c_r \end{bmatrix} \in \mathcal{B}(\mathbb{C}^r, \mathcal{Y})$$

have the coefficients

$$a_{ij} = \frac{\sqrt{\sigma_j}}{\sqrt{\sigma_i}} \left\langle \widetilde{v}_i, \tfrac{d}{d\xi}\widetilde{v}_j \right\rangle_{L^\infty(\mathbb{R}_+;\mathcal{Y}),L^1(\mathbb{R}_+;\mathcal{Y})} \in \mathbb{C}, \tag{4.4a}$$
$$b_i = \sqrt{\sigma_i}\, \langle \cdot, \widetilde{u}_i(0) \rangle_{\mathcal{U}} \in \mathcal{B}(\mathcal{U}, \mathbb{C}), \tag{4.4b}$$
$$c_j = \frac{1}{\sqrt{\sigma_j}} \widetilde{v}_j(0) \in \mathcal{Y}. \tag{4.4c}$$

(ii) The output-normalized truncation is defined analogously with

$$a_{ij} = \left\langle \widetilde{v}_i, \tfrac{d}{d\xi}\widetilde{v}_j \right\rangle_{L^\infty(\mathbb{R}_+;\mathcal{Y}),L^1(\mathbb{R}_+;\mathcal{Y})} \in \mathbb{C},$$
$$b_i = \sigma_j\, \langle \cdot, \widetilde{u}_i(0) \rangle_{\mathcal{U}} \in \mathcal{B}(\mathcal{U}, \mathbb{C}),$$
$$c_j = \widetilde{v}_j(0) \in \mathcal{Y}.$$

Remark 4.3. a) The well-definition of the above dual products and evaluations is guaranteed by the fact that $\widetilde{u}_j \in W^{1,1}(\mathbb{R}_-;\mathcal{U})$, $\widetilde{v}_j \in W^{1,1}(\mathbb{R}_+;\mathcal{Y})$ [8, Theorem 5.2.2]. Furthermore, in the case where the function h in (2.3) additionally fulfills $h \in L^2(\mathbb{R}_+, \mathcal{B}(\mathcal{U};\mathcal{Y}))$, the Schmidt vectors fulfill $\widetilde{u}_j \in W^{1,2}(\mathbb{R}_-;\mathcal{U})$, $\widetilde{v}_j \in W^{1,2}(\mathbb{R}_+;\mathcal{Y})$ [8, Lemma 5.2.12]. In this case, the entries of A_r are indeed inner products in L^2.

b) In [8,9], only the output normalized truncation is used as reduced order model. Note that the output normalized and the balanced truncation model are related by a state space transformation with $\sqrt{\sigma_i}$. In particular, these two models have the same transfer function (and thus also the same input-output mapping).

c) The Schmidt pairs of the Hankel operator are, even for finite-dimensional systems, quite impossible to compute. Instead, one performs coordinate transformations

$$\begin{aligned} A_b &= TAT^+ := (\Sigma^{-1/2}V^*S^*)A(RU\Sigma^{-1/2}), \\ B_b &= TB := (\Sigma^{-1/2}V^*S^*)B, \\ C_b &= CT^+ := C(RU\Sigma^{-1/2}). \end{aligned} \tag{4.6}$$

where, Σ, S, R, U and V can be determined from the Gramians of the system (2.1), see [1, Sec. 7.3]. It is an aim of this article to consider such transformations for infinite-dimensional systems of the class described in Section 2.

d) The Gramians of the truncated balanced realization (4.3) are given by

$$P_r = Q_r = \Sigma_r = \operatorname{diag}(\sigma_n)_{n=1,\dots,r},$$

whence the Hankel singular values of the truncated balanced realization (4.3) are given by $\sigma_1, \ldots, \sigma_r$. However, the Hankel operator of (4.3) does in general not coincide with the truncation of a singular value decomposition of the Hankel operator of the original system (2.1).

Theorem 4.2. *[8, Thm 5.0.2] With the prerequisites and notation of Definition* 4.2*, there holds that* (4.3) *is a minimal, bounded well-posed linear system on* $(\mathcal{U}, \mathbb{C}^r, \mathcal{Y})$*. Moreover, the transfer functions* $G, G_r : \mathbb{C}_+ \to \mathcal{B}(\mathcal{U}; \mathcal{Y})$ *of* (4.3) *and* (2.1) *fulfill*

$$\|G - G_r\|_{\mathcal{H}_\infty} \leqslant 2 \sum_{\{n>r \mid \sigma_n \neq \sigma_k \forall k<n\}}^{\infty} \sigma_n. \tag{4.7}$$

It follows from (d) that the error bound (4.7) can be used to estimate the expression $\|y - y_r\|_{L^2(\mathbb{R};\mathcal{Y})}$, where y and y_r are the respective outputs of (2.1) and (4.3) with same input $u \in L^2(\mathbb{R};\mathcal{U})$.

5. Main results

Throughout the rest of the article we will work with the following setup: The quadruple $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is a bounded well-posed linear system generated by the operators (A, B, C, D) which satisfy **S1** through **S6**. Moreover, X_R and X_S are Hilbert spaces, and $R \in \mathcal{B}(X_R, X)$, $S \in \mathcal{B}(X_S, X)$ are operators such that the controllability and observability Gramians satisfy

$$P = \mathfrak{B}\mathfrak{B}^* = RR^* \qquad \text{and} \qquad Q = \mathfrak{C}^*\mathfrak{C} = SS^*. \tag{5.1}$$

Of course, these factors might, for instance, be $R = \mathfrak{B}$, $S = \mathfrak{C}^*$, or $R = P^{1/2}$, $S = Q^{1/2}$. Note that the so-called *ADI method* [17] directly provides factors R and S of the Gramians. The following results are about the construction of balanced realizations on the basis of R and S. First we show that the singular values of the operator S^*R are the Hankel singular values. Thereafter, we construct balanced realizations and truncated balanced realization by using a singular value decomposition of S^*R. We further show that we can associate, in a certain sense, infinite matrices to the generators of a balanced realization.

Theorem 5.1. *Let a system* (2.1) *with properties* **S1**–**S6** *be given. Further, let* X_R, X_S *be Hilbert spaces and let* $R \in \mathcal{B}(X_R; X)$*,* $S \in \mathcal{B}(X_S; X)$ *be operators such that the Gramians of* (2.1) *satisfy* (5.1)*. Then the following is true for the Hankel operator* $\mathfrak{H} = \mathfrak{C}\mathfrak{B}$ *of* (2.1)*:*

a) There exist unitary operators

$$\begin{aligned} \mathfrak{V} : &\quad \overline{\operatorname{ran} \mathfrak{H}} \subset L^2(\mathbb{R}_+; \mathcal{Y}) \to \overline{\operatorname{ran} S^*R} \subset X_S, \\ \mathfrak{U} : &\quad (\ker \mathfrak{H})^\perp \subset L^2(\mathbb{R}_-; \mathcal{U}) \to (\ker S^*R)^\perp \subset X_R, \end{aligned}$$

such that

$$\mathfrak{V}^*\mathfrak{H}\mathfrak{U}x_R = S^*Rx_R \text{ for all } x_R \in (\ker S^*R)^\perp. \tag{5.2}$$

b) $S^*R \in \mathcal{B}(X_R; X_S)$ *is compact, if, and only if,* $\mathfrak{H}$ *is compact. In this case, the singular values of* S^*R *are the Hankel singular values in* Σ*.*

c) $S^*R \in \mathcal{B}(X_R; X_S)$ *is nuclear, if, and only if,* $\mathfrak{H}$ *is nuclear.*

The following theorem shows that the singular value decomposition of S^*R can be utilized to construct balanced realizations. Thereafter, we consider the generators of balanced realizations.

Theorem 5.2. *Let a system* (2.1) *with properties* **S1**–**S6** *and* **H4** *be given. Let* $R \in \mathcal{B}(X_R; X)$ *and* $S \in \mathcal{B}(X_S; X)$ *be as in* (5.1) *and, with the notation of* (1.1) *and* (1.2)*, let*

$$S^*R = V\Sigma U^* \tag{5.3}$$

be a singular value decomposition of the operator $S^*R \in \mathcal{B}(X_R; X_S)$*. Then the operators*

$$\begin{aligned} \mathcal{T} : X &\to \ell_2, & \mathcal{T}^+ : \Sigma\ell_2 \subset \ell_2 &\to X, \\ x &\mapsto V^*S^*x, & x &\mapsto RU\Sigma^{-1}x \end{aligned} \tag{5.4}$$

are well-defined, and the following assertions hold true:

a) *There exist a constant* $c > 0$ *such that, for all* $x \in \Sigma\ell_2$*,* $u \in L^2(\mathbb{R}_-; \mathcal{U})$ *and* $t \in \mathbb{R}_+$*, holds*

$$\|\mathcal{T}\mathfrak{A}(t)\mathcal{T}^+x\|_{\ell_2} \leqslant c\ \|x\|_{\ell_2}, \quad \|\mathcal{T}\mathfrak{B}u\|_{\ell_2} \leqslant c\ \|u\|_{L^2(\mathbb{R}_-;\mathcal{U})},$$
$$\|\mathfrak{C}\mathcal{T}^+x\|_{L^2(\mathbb{R}_+;\mathcal{Y})} \leqslant c\ \|x\|_{\ell_2}.$$

b) *With the unique continuous extensions*

$$\overline{\mathcal{T}\mathfrak{A}\mathcal{T}^+} : \mathbb{R}_+ \to \mathcal{B}(\ell_2),\ t \mapsto \overline{\mathcal{T}\mathfrak{A}(t)\mathcal{T}^+}, \quad \text{and} \quad \overline{\mathfrak{C}T^+} \in \mathcal{B}(\ell_2; L^2(\mathbb{R}_+; \mathcal{Y})),$$

the quadruple

$$(\mathfrak{A}_o, \mathfrak{B}_o, \mathfrak{C}_o, \mathfrak{D}) := (\overline{\mathcal{T}\mathfrak{A}\mathcal{T}^+}, \mathcal{T}\mathfrak{B}, \overline{\mathfrak{C}\mathcal{T}^+}, \mathfrak{D}) \tag{5.5}$$

is a minimal, output-normalized, bounded well-posed linear system on ℓ_2*, which is unitarily similar to the exactly observable shift realization of* $\mathfrak{D}$ *on* $\overline{\operatorname{ran}\mathfrak{H}}$*. We call this the output normalized realization of* $\mathfrak{D}$ *on* ℓ_2*.*

If, in addition, **H3** and **R1** are satisfied, a representation of the generators can essentially be calculated via these transformations T and T^+, similarly to (4.6).

Theorem 5.3. *Assume that in Theorem* 5.2*, the assumptions* **H3** *and* **R1** *hold in addition. Then the following is true for the generators* (A_o, B_o, C_o, D) *of* (5.5)*:*

a) *The space* $\mathcal{Z} := \mathcal{T}\mathfrak{B}W_0^{1,2}(\mathbb{R}_-; \mathcal{U})$ *is a subset of* $\Sigma\ell_2$ *and a core for* A_o*. Moreover,* $A\mathcal{T}^+\mathcal{Z} \subset \operatorname{ran} R$ *and* $T^+\mathcal{Z} \subset \operatorname{dom} C$*;*

b) *For all* $z \in \mathcal{Z}$ *holds*

$$A_o z = \mathcal{T}\widetilde{A}\pi_{(\ker S^*)^\perp}\mathcal{T}^+ z, \tag{5.6}$$
$$C_o = C\mathcal{T}^+ z. \tag{5.7}$$

c) The adjoint operator $(\mathcal{T}|_{\overline{M}})^*$ *of* $\mathcal{T}|_{\overline{M}}$ *is given by* $\pi_{\overline{M}}SV$ *and maps* $\operatorname{dom} A_o^*$ *into* $\operatorname{dom}\widetilde{A}$. *Thus, the operator* $\mathcal{T}|_{\overline{M}}$ *has a continuous extension* $\mathcal{T}_{-1} : (\operatorname{dom}\widetilde{A})' \to (\operatorname{dom} A_o^*)'$ *defined by*

$$\langle \mathcal{T}_{-1}x', (y_n)\rangle_{(\operatorname{dom} A_o^*)',\operatorname{dom} A_o^*} = \langle x', \pi_{\overline{M}}SV(y_n)\rangle_{\ell_2}, \tag{5.8}$$

for all $x' \in (\operatorname{dom}\widetilde{A}^*)'$ *and* $(y_n) \in \operatorname{dom} A_o^*$.

d) With the extension from c) holds

$$A_o|_{\ell_2}x = \mathcal{T}_{-1}\widetilde{A}|_{\overline{M}}\pi_{(\ker S^*)^\perp}\mathcal{T}^+z, \qquad \forall x \in \Sigma\ell_2 \tag{5.9}$$

$$B_o = \mathcal{T}_{-1}\widetilde{B}. \tag{5.10}$$

Moreover, A_o *and* $(A_o)_{-1}$ *are obtained by taking the closures of the respective operators above and the set* $(\mathcal{T}\operatorname{dom} C) \cap \Sigma\ell_2$ *contains a core of* A_o.

Remark 5.1. In fact, the formulas in *b)* are valid on larger sets than $\mathcal{Z}$. Namely (5.6) holds on $(\mathcal{T}\operatorname{dom} A) \cap \Sigma\ell_2$, and (5.7) on $(\mathcal{T}\operatorname{dom} C) \cap \Sigma\ell_2$.

Theorem 5.4. *Let a system* (2.1) *with properties* **S1**–**S6** *and* **H4** *be given. With* $R \in \mathcal{B}(X_R; X)$ *and* $S \in \mathcal{B}(X_S; X)$ *as in* (5.1) *and the notation of* (1.1), (1.2), *let* (5.3) *be a singular value decomposition of the operator* $S^*R \in \mathcal{B}(X_R; X_S)$. *Then the mappings*

$$\begin{aligned} T :\ \operatorname{ran} R \subset X &\to \ell_2, & T^+ :\ \Sigma^{1/2}\ell_2 \subset \ell_2 &\to X, \\ x &\mapsto \Sigma^{-1/2}V^*S^*x, & x &\mapsto RU\Sigma^{-1/2}x \end{aligned} \tag{5.11}$$

are well-defined, and the following assertions are true:

a) $\mathfrak{A}(t)\operatorname{ran} R \subset \operatorname{ran} R$ *for all* $t \in \mathbb{R}_+$ *and* $\operatorname{ran}\mathfrak{B} = \operatorname{ran} R$;

b) There exists a constant $c > 0$ *such that, for all* $x \in \Sigma^{1/2}\ell_2$, $u \in L^2(\mathbb{R}_-; \mathcal{U})$ *and* $t \in \mathbb{R}_+$, *holds*

$$\|T\mathfrak{A}(t)T^+x\|_{\ell_2} \leqslant c\ \|x\|_{\ell_2}, \quad \|T\mathfrak{B}u\|_{\ell_2} \leqslant c\ \|u\|_{L^2(\mathbb{R}_-;\mathcal{U})},$$
$$\|\mathfrak{C}T^+x\|_{L^2(\mathbb{R}_+;\mathcal{Y})} \leqslant c\ \|x\|_{\ell_2}.$$

c) With the unique continuous extensions

$$\overline{T\mathfrak{A}T^+} : \mathbb{R}_+ \to \mathcal{B}(\ell_2),\ t \mapsto \overline{T\mathfrak{A}(t)T^+}, \quad \text{and} \quad \overline{\mathfrak{C}T^+} \in \mathcal{B}(\ell_2; L^2(\mathbb{R}_+; \mathcal{Y})),$$

the quadruple

$$(\mathfrak{A}_b, \mathfrak{B}_b, \mathfrak{C}_b, \mathfrak{D}) := (\overline{T\mathfrak{A}T^+}, T\mathfrak{B}, \overline{\mathfrak{C}T^+}, \mathfrak{D}) \tag{5.12}$$

forms a minimal and balanced, bounded well-posed linear system on $(\mathcal{U}, \ell_2, \mathcal{Y})$.

Theorem 5.5. *Assume that in Theorem* 5.4 *the assumptions* **S1**–**S6**, **H3** *and* **R1** *hold. Then the following is true for the generators* (A_b, B_b, C_b, D_b) *of the balanced realization* (5.12):

a) The space $Z := T\mathfrak{B}W_0^{1,2}(\mathbb{R}_-; \mathcal{U})$ *is a subset of* $\Sigma^{1/2}\ell_2$ *and a core for* A_b. *Moreover,* $AT^+Z \subset \operatorname{ran} R$ *and* $T^+Z \subset \operatorname{dom} C$;

b) For all $x \in Z$ holds

$$\begin{aligned} A_b x &= T\widetilde{A}\pi_{(\ker S^*)^\perp} T^+ x, \\ C_b x &= C T^+ x. \end{aligned}$$

c) There exists a space $\widetilde{Z} \subset \Sigma^{1/2}\ell_2 \cap \operatorname{dom} A_b^*$, which is a core for A_b^*, such that the adjoint

$$\begin{aligned} (T|_{\overline{M}})^* : \quad \Sigma^{1/2}\ell_2 \subset \ell_2 &\to X, \\ x &\mapsto \pi_{\overline{M}} S V \Sigma^{-1/2} x \end{aligned}$$

fulfills $(T|_{\overline{M}})^* \widetilde{Z} \subset \operatorname{dom} \widetilde{A}^*$, with $\widetilde{A}$ as in Lemma 3.1.

d) For all $z \in \widetilde{Z}$, $u \in \mathcal{U}$ holds

$$\langle B_b u, x \rangle_{(\operatorname{dom} A_b^*)', \operatorname{dom} A_b^*} = \langle Bu, (T|_{\overline{M}})^* x \rangle_{(\operatorname{dom} A^*)', \operatorname{dom} A^*}. \tag{5.13}$$

That is, $B_b u$ is obtained by continuous extension of this functional to $\operatorname{dom} A_b^*$.

Remark 5.2. Relation (5.13) is a generalization of the expression $B_b = TB$, in the sense of Lemma 1.2.

Now we present that, in a certain sense, the generators of balanced realizations can be regarded as infinite matrices.

Theorem 5.6. *Under the prerequisites of Theorem 5.5, let (A_b, B_b, C_b, D) be the generators of the balanced realization (5.12). There exists a space $Z_b \hookrightarrow \ell_2$ such that the following holds true:*

a) For all $i \in \mathbb{N}$ the canonical unit vector $e_i = (\delta_{i,1}, \delta_{i,2}, \ldots) \in \ell_2$ is an element of Z_b;

b) $A_b|_{\ell_2} e_i \in Z_b'$ for all i, $\operatorname{ran} B_b \subset Z_b'$ and $Z_b \subset \operatorname{dom} C_b$;

c) For the coefficients a_{ij}, b_i, c_j of the truncated balanced realization in Definition 4.2 (i) holds

$$\begin{aligned} a_{ij} &= \langle A_b e_j, e_i \rangle_{Z_b', Z_b} && \in \mathbb{C}, \\ b_i(\cdot) &= \langle B_b \cdot, e_i \rangle_{Z_b', Z_b} = \big\langle u, (B_b')_L e_i \big\rangle_{\mathcal{U}} && \in \mathcal{B}(\mathcal{U}; \mathbb{C}), \\ c_j &= C_b e_j && \in \mathcal{Y}, \end{aligned}$$

where $(B_b')_L$ is the Lebesgue extension of B_b'.

An immediate consequence of this is that a truncated balanced system can indeed be obtained by truncating the generators (A_b, B_b, C_b, D_b) of a balanced system.

Theorem 5.7. *Under the prerequisites of Theorem 5.3, an analogous statement to Theorem 5.6 holds, when a_{ij}, b_i, c_j are the coefficients of the output normalized truncation in Definition 4.2 (ii) and (A_b, B_b, C_b, D) are replaced with the generators (A_o, B_o, C_o, D) of the output normalized realization from Theorem 5.3.*

Remark 5.3. a) Theorem 5.6 and 5.7 also hold when $h \in L^2(\mathbb{R}_+; \mathcal{B}(\mathcal{U}, \mathcal{Y})$. The corresponding spaces Z_b, respectively Z_o, in b) of these theorems just have to be adapted, cf. Remark 9.1. In fact, in this case it is possible to choose $Z_b = \Sigma^{1/2}\ell_2$, $Z_b' = \Sigma^{-1/2}\ell_2$ with $\langle \cdot, \cdot \rangle_{Z_b', Z_b} := \langle \Sigma^{1/2}\cdot, \Sigma^{-1/2}\cdot \rangle_{\ell_2}$ in Theorem 5.6 and $Z_o = Z_o' = \ell_2$ in Theorem 5.7.

b) For calculating the balanced truncation, it is somewhat easier to determine the normalized truncation first and then the balanced one via the finite-dimensional state space transformation mentioned in Remark 4.1 b). Theorem 5.6 shows that this results in the same reduced model as truncating the balanced realization on ℓ_2 would.

Theorem 5.8. *Under the prerequisites of Theorem* 5.5*, let*

$$S^*R = \sum_{n=1}^{\infty} \sigma_n \langle x, u_n \rangle_X v_n,$$

be a singular value decomposition of S^*R. Then there exists a space $Z \hookrightarrow X$ such that the following holds true:

a) $\operatorname{ran} B \subset Z'$ *and* $Z \subset \operatorname{dom} C$.
b) $Sv_i \in Z$ *and* $A|_X Ru_i \in Z'$ *for all* $i \in \mathbb{N}$.
c) For the coefficients a_{ij}, b_i, c_j *of the balanced and truncated realization from Definition* 4.2 *holds:*

$$\begin{aligned} a_{ij} &= \frac{1}{\sqrt{\sigma_i \sigma_j}} \langle A|_X Ru_j, Sv_i \rangle_{Z',Z} && \in \mathbb{C}, \\ b_i(\cdot) &= \frac{1}{\sqrt{\sigma_i}} \langle B\cdot, Sv_i \rangle_{Z',Z} = \left\langle \cdot, (B')_L Sv_i \right\rangle_{\mathcal{U}} && \in \mathcal{B}(\mathcal{U}, \mathbb{C}) \\ c_j &= \frac{1}{\sqrt{\sigma_i}} CRu_j && \in \mathcal{Y}, \end{aligned}$$

where $(B')_L$ *is the Lebesgue extension of* B'.

The forthcoming sections are devoted to the proofs of the main results.

6. Proof of Theorem 5.1

With the assumptions and notation of Theorem 5.1, there holds

$$\operatorname{ran} R = \operatorname{ran} \mathfrak{B}, \qquad \operatorname{ran} \mathfrak{C}^* = \operatorname{ran} S, \tag{6.1a}$$

$$\ker \mathfrak{B}^* = \ker R^*, \qquad \ker S^* = \ker \mathfrak{C}. \tag{6.1b}$$

The equations in (6.1a) are consequences of the fact that the operator square roots fulfill

$$\operatorname{ran} R = \operatorname{ran} \sqrt{RR^*} = \operatorname{ran} \sqrt{\mathfrak{B}\mathfrak{B}^*} = \operatorname{ran} \mathfrak{B} \qquad \text{and}$$

$$\operatorname{ran} \mathfrak{C}^* = \operatorname{ran} \sqrt{\mathfrak{C}^*\mathfrak{C}} = \operatorname{ran} \sqrt{SS^*} = \operatorname{ran} S,$$

see e.g. [10, pp. 334-336]. The remaining assertions (6.1b) follow by regarding the orthogonal complements in (6.1a). With this, the restricted operators

$$\begin{aligned} R: &\quad (\ker R)^\perp \subset X_R \to \overline{\operatorname{ran}\mathfrak{B}}, \quad S: \quad (\ker S)^\perp \subset X_S \to (\ker \mathfrak{C})^\perp, \\ \mathfrak{B}: &\quad (\ker \mathfrak{B})^\perp \subset L^2(\mathbb{R}_-;\mathcal{U}) \to \overline{\operatorname{ran}\mathfrak{B}}, \quad \mathfrak{C}: \quad (\ker \mathfrak{C})^\perp \subset X \to \overline{\operatorname{ran}\mathfrak{C}}. \end{aligned}$$

are injective and have dense range. We denote their inverses (and adjoints of their inverses) by R^{-1}, $\mathfrak{B}^{-1}$, S^{-1} and $\mathfrak{C}^{-1}$ (R^{-*}, $\mathfrak{B}^{-*}$, S^{-*} and $\mathfrak{C}^{-*}$). Note that for any injective, closed and densely defined operator T with dense range holds $(\operatorname{dom} T^{-1})^* = \operatorname{ran} T^*$ and hence $T^{-*} := (T^{-1})^* = (T^*)^{-1}$, see e.g. [18, Lemma 3.5.2].

Lemma 6.1. *The mappings*

$$\mathfrak{V} : \overline{\operatorname{ran} S^*R} \to \overline{\operatorname{ran}\mathfrak{H}}, \qquad \mathfrak{V} := \overline{\mathfrak{C}S^{-*}}|_{\overline{\operatorname{ran} S^*R}}, \tag{6.2}$$

where $\overline{\mathfrak{C}S^{-*}}$ *is the continuous extension of* $\mathfrak{C}S^{-*}|_{\operatorname{ran} S^*R}$ *with respect to the norms of* X_S *and* $L^2(\mathbb{R}_+;\mathcal{Y})$*, and*

$$\mathfrak{U} : (\ker S^*R)^\perp \to (\ker \mathfrak{H})^\perp, \qquad \mathfrak{U} := \mathfrak{B}^{-1}R|_{(\ker S^*R)^\perp}. \tag{6.3}$$

are unitary with inverses $\mathfrak{V}^* = \overline{S^*\mathfrak{C}^{-1}}$ *and* $\mathfrak{U}^* = R^{-1}\mathfrak{B}$ *respectively.*

Proof. From the fact that $\operatorname{ran} R = \operatorname{ran}\mathfrak{B}$ and

$$\|\mathfrak{C}S^{-*}x\|^2_{L^2(\mathbb{R}_+;\mathcal{Y})} = \langle S^{-*}x, \mathfrak{C}^*\mathfrak{C}S^{-*}x\rangle_X = \langle S^{-*}x, Sx\rangle_X = \|x\|^2_{X_S}$$

for all $x \in \operatorname{ran} S^*$, we deduce that $\mathfrak{C}S^{-*} : \operatorname{ran} S^*R \to \overline{\operatorname{ran}\mathfrak{H}}$ is an isometry with dense range and inverse $S^*\mathfrak{C}^{-1}$. Therefore, it can be extended to a unitary operator $\mathfrak{U}$ between the closures of these two spaces. Analogously, we can deduce that the concatenation $R^*\mathfrak{B}^{-*} : \operatorname{ran}\mathfrak{B}^*\mathfrak{C}^* \to \operatorname{ran} R^*S$ satisfies

$$\|R^*\mathfrak{B}^{-*}x\|_{X_R} = \|x\|_{L^2(\mathbb{R}_-;\mathcal{U})} \quad \forall x \in \operatorname{ran}\mathfrak{B}^*,$$

and has a unitary extension to the closures $\mathfrak{U}^* : (\ker\mathfrak{H})^\perp \to (\ker S^*R)^\perp$. Furthermore, because of (5.1), the identity $R^*\mathfrak{B}^{-*}x = R^{-1}\mathfrak{B}x$ holds for all $x \in (\ker\mathfrak{H})^\perp$. But the operator $R^{-1}\mathfrak{B}|_{(\ker\mathfrak{H})^\perp}$ is defined on the complete space $(\ker\mathfrak{H})^\perp$ and it is closed because R^{-1} is closed. By the closed graph theorem it is continuous and hence, it must be equal to the unique unitary extension of $R^*\mathfrak{B}^{-*}$. This means that both $R^{-1}\mathfrak{B}|_{(\ker\mathfrak{H})^\perp}$ and its inverse $\mathfrak{B}^{-1}R|_{(\ker S^*R)^\perp}$ are bounded with norm 1. □

Proof of Theorem 5.1. The equation

$$\begin{aligned} \mathfrak{H}|_{(\ker\mathfrak{H})^\perp} &= \mathfrak{C}\pi_{(\ker S^*)^\perp}\mathfrak{B}|_{(\ker\mathfrak{H})^\perp} \\ &= \mathfrak{C}S^{-*}S^*RR^{-1}\mathfrak{B}|_{(\ker\mathfrak{H})^\perp} \\ &= \mathfrak{V}(S^*R)|_{(\ker S^*R)^\perp}\mathfrak{U}^*|_{(\ker\mathfrak{H})^\perp} \end{aligned}$$

shows (5.2). Hence the compactness claim holds, because $\mathfrak{U}$ and $\mathfrak{V}$ are unitary. In particular, we can find a singular value decomposition of the form (5.3) if

the Hankel operator is compact. Now the equalities

$$(S^*R)(S^*R)^*|_{\overline{\operatorname{ran} S^*R}} = \mathfrak{V}^*\mathfrak{H}\mathfrak{H}^*\mathfrak{V}|_{\overline{\operatorname{ran} S^*R}} \quad \text{and}$$
$$(S^*R)^*(S^*R)|_{(\ker S^*R)^\perp} = \mathfrak{U}^*\mathfrak{H}^*\mathfrak{H}\mathfrak{U}|_{(\ker S^*R)^\perp}$$

show that v_i is an eigenvector of $S^*R(S^*R)^*$ to the eigenvalue $\sigma_i^2 > 0$ if and only if $\widetilde{v}_i := \mathfrak{V}v$ is an eigenvector of $\mathfrak{H}\mathfrak{H}^*$ corresponding to the same eigenvalue and, analogously, u_i is an eigenvector of $(S^*R)^*S^*R$ if and only if $\widetilde{u}_i := \mathfrak{U}u_i$ is an eigenvector of $\mathfrak{H}^*\mathfrak{H}$. Hence it follows that the singular values of S^*R and $\mathfrak{H}$ are equal. In particular,

$$\mathfrak{H}u = \sum_{i=i}^{\infty} \widetilde{v}_i \sigma_i \langle u, \widetilde{u}_i \rangle \quad \forall u \in L^2(\mathbb{R}_-; \mathcal{U})$$

is a singular value decomposition of $\mathfrak{H}$. □

7. Proof of Theorem 5.2 and 5.3

Lemma 7.1. *Define* $M := \pi_{(\ker S^*)^\perp} \operatorname{ran} R = \pi_{(\ker \mathfrak{C}^*)^\perp} \operatorname{ran} \mathfrak{B}$ *as in Theorem* 3.1. *The mapping* $\mathcal{T}|_M : M \to \Sigma\ell_2$ *is an isomorphism with inverse given by*

$$\pi_{(\ker S^*)^\perp} \mathcal{T}^+(x_n) = S^{-*}V(x_n) \quad \forall (x_n) \in \Sigma\ell_2 \tag{7.1}$$

Proof. Using equation (5.3) and $VV^* = \pi_{\overline{\operatorname{ran} S^*R}}$, it is not hard to see that $\mathcal{T}|_M = V^*S^*$ is an isomorphism between the claimed spaces with inverse $S^{-*}V$. The important part here is that the correct spaces were chosen. The singular value decomposition also shows immediately that V^*S^* is the left inverse of $\pi_{(\ker S^*)^\perp} RU\Sigma^{-1} = \pi_{(\ker S^*)^\perp}\mathcal{T}$ on $\Sigma\ell_2$. To prove that it is also a right inverse we calculate for given $y = \pi_{(\ker S^*)^\perp} Rx$ with $x \in X_R$

$$\begin{aligned}\pi_{(\ker S^*)^\perp} RU\Sigma^{-1}V^*S^*y &= \pi_{(\ker S^*)^\perp} RU\Sigma^{-1}V^*S^*Rx = \pi_{(\ker S^*)^\perp} RUU^*x \\ &= \pi_{(\ker S^*)^\perp} R\pi_{(\ker S^*R)^\perp} x = \pi_{(\ker S^*)^\perp} Rx = y.\end{aligned}$$

□

Proof of Theorem 5.2. We are going to show that, with $\mathfrak{V}$ from Lemma 6.1, the mapping

$$\widetilde{V} : \ell_2 \to \overline{\operatorname{ran} \mathfrak{H}}, \qquad (x_n) \mapsto \mathfrak{V}V(x_n) = \sum_{n=1}^{\infty} x_n \mathfrak{V}v_n,$$

which is unitary with inverse $\widetilde{V}^* = V^*\mathfrak{V}^*$, transforms the shift realization (4.2) into the system (5.5). Well-posedness of the latter follows immediately from Lemma 2.1. First note that, due to $\ker \mathfrak{C} = \ker S^*$ and Definition 2.1 (iii), we have the following expression for all $x \in \operatorname{ran} S^*R$

$$\begin{aligned}S^*\mathfrak{A}(t)S^{-*}x &= S^*\pi_{(\ker \mathfrak{C})^\perp}\mathfrak{A}(t)\pi_{(\ker \mathfrak{C})^\perp}S^{-*}x = S^*\mathfrak{C}^{-1}\mathfrak{C}\mathfrak{A}(t)\mathfrak{C}^{-1}\mathfrak{C}S^{-*}x = \\ &= \mathfrak{V}^*\mathfrak{C}\mathfrak{A}(t)\mathfrak{C}^{-1}\mathfrak{V}x = \mathfrak{V}^*\tau_+^t|_{\overline{\operatorname{ran} \mathfrak{H}}}\mathfrak{V}x.\end{aligned}$$

Furthermore, for all $x \in \operatorname{ran} \mathfrak{H}$ we have $\mathfrak{V}^* x \in \operatorname{ran} S^* R$ and hence $V^* \mathfrak{V}^* x \in \Sigma \ell_2$. By Lemma 7.1 we can substitute

$$\begin{aligned}\mathfrak{V} V \mathcal{T} \mathfrak{A}(t) \mathcal{T}^+ V^* \mathfrak{V}^* x &= \mathfrak{V} V V^* S^* \mathfrak{A}(t) \pi_{(\ker S^*)^\perp} R U \Sigma^{-1} V^* \mathfrak{V}^* x \\ &= \mathfrak{V} V V^* S^* \mathfrak{A}(t) S^{-*} V V^* \mathfrak{V}^* x = \\ &= \mathfrak{V} \pi_{\overline{\operatorname{ran} S^* R}} S^* \mathfrak{A}(t) S^{-*} \pi_{\overline{\operatorname{ran} S^* R}} \mathfrak{V}^* x \\ &= \mathfrak{V} \pi_{\overline{\operatorname{ran} S^* R}} \mathfrak{V}^* \tau_+^t \mathfrak{V} \pi_{\overline{\operatorname{ran} S^* R}} \mathfrak{V}^* x = \tau_+^t x,\end{aligned}$$

and by continuous extension it follows that this formula holds on the closure of $\operatorname{ran} \mathfrak{H}$. Furthermore, one gets

$$\begin{aligned}\mathfrak{C} \mathcal{T}^+ V^* \mathfrak{V}^* x &= \mathfrak{C} R U \Sigma^{-1} V^* \mathfrak{V}^* x = \mathfrak{C} \pi_{(\ker S^*)^\perp} R U \Sigma^{-1} V^* \mathfrak{V}^* x \\ &= \mathfrak{C} S^{-*} V V^* \mathfrak{V}^* x = \mathfrak{V} \pi_{\overline{\operatorname{ran} S^* R}} \mathfrak{V}^* x = x.\end{aligned}$$

Again, continuous extension yields that $\mathfrak{C} R U \Sigma^{-1}$ is similar to $\mathrm{id}_{\overline{\operatorname{ran} \mathfrak{H}}}$ via the unitary transformation $V^* \mathfrak{V}^*$. Finally,

$$\mathfrak{V} V \mathcal{T} \mathfrak{B} = \mathfrak{V} V V^* S^* \mathfrak{B} = \overline{\mathfrak{C} S^{-*}} S^* \mathfrak{B} = \overline{\mathfrak{C} S^{-*}} \pi_{\overline{\operatorname{ran} S^* R}} S^* \mathfrak{B} = \mathfrak{C} \mathfrak{B} = \mathfrak{H},$$

completes the proof of the asserted similarity and well-posedness and applying Lemma 2.1 proves the Theorem. □

Corollary 7.2. *If in Theorem* 5.2 *the assumptions* **H3** *and* **R1** *hold as well, then the generators of* (5.5) *are*

$$\operatorname{dom} A_o = \left\{ (x_n) \in \ell_2 : \sum_{n=1}^{\infty} x_n \widetilde{v}_n \in W^{1,2}(\mathbb{R}_+; \mathcal{Y}) \right\}, \tag{7.2a}$$

$$A_o(x_n) = \widetilde{V}^* \tfrac{d}{d\xi} \widetilde{V}(x_n), \tag{7.2b}$$

$$\operatorname{dom} A_o^* = \left\{ (x_n) \in \ell_2 : \sum_{n=1}^{\infty} x_n \widetilde{v}_n \in \pi_{\overline{\operatorname{ran} \mathfrak{H}}} W_0^{1,2}(\mathbb{R}_+; \mathcal{Y}) \right\}, \tag{7.3a}$$

$$A_o^*(x_n) = -\widetilde{V}^* \tfrac{d}{d\xi} y \tag{7.3b}$$

for any $y \in W_0^{1,2}(\mathbb{R}_+; \mathcal{Y})$ *with* $\pi_{\overline{\operatorname{ran} \mathfrak{H}}} y = \widetilde{V}(x_n)$.

For each $u \in \mathcal{U}$, *the image* $B_o u$ *is an element of* $(\operatorname{dom} A_o^*)'$ *acting as follows:*

$$(B_o u)(x_n) = \langle hu, y \rangle_{L^1(\mathbb{R}_+; \mathcal{Y}), L^\infty(\mathbb{R}_+; \mathcal{Y})}, \tag{7.4}$$

for any $y \in W_0^{1,2}(\mathbb{R}_+; \mathcal{Y})$ *with* $\pi_{\overline{\operatorname{ran} \mathfrak{H}}} y = \widetilde{V}(x_n)$.

Furthermore holds

$$\operatorname{dom} C_o = \left\{ (x_n) \in \ell_2 : \sum_{n=1}^{\infty} x_n \widetilde{v}_n \text{ has a Lebesgue point at zero.} \right\},$$

$$C_o(x_n) = \lim_{t \to 0} \frac{1}{t} \int_0^t \sum_{n=1}^{\infty} x_n \widetilde{v}_n(\tau) d\tau. \tag{7.5}$$

All the series here are limits in the $L^2(\mathbb{R}_+; \mathcal{Y})$ *norm.*

Proof. Since we have shown in the proof of Theorem 5.2 that (5.5) is unitary simliar to (4.2) via $\widetilde{V}$, the generators of (5.5) are also obtained via unitary transformation of the operators in Lemma 4.1 (ii). Note that

$$\operatorname{dom} A_o = \left\{ (x_n) \in \ell_2 \;\middle|\; \widetilde{V}(x_n) \in (\overline{\operatorname{ran} \mathfrak{H}} \cap W^{1,2}(\mathbb{R}_+; \mathcal{Y}) \right\},$$

which becomes (7.2a) because $\widetilde{V}(x_n)$ is always in $\overline{\operatorname{ran} \mathfrak{H}}$. □

The space $\mathcal{Z}$ introduced in Theorem 5.3 *a)* will play an important role in the sequel, because it is a core for A_o, as we show in the next proposition.

Proposition 7.3. *The space* $\mathcal{Z} := \mathcal{T}\mathfrak{B}W_0^{1,2}(\mathbb{R}_-; \mathcal{U})$ *is a core for* A_o *and*

$$A_o z = \mathcal{T}\widetilde{A}\pi_{(\ker S^*)^\perp}\mathcal{T}^+ z \qquad \forall z \in \mathcal{Z}, \tag{7.6}$$

where $\widetilde{A}$ *is as in Lemma* 3.1. *Consequently, the graph of* A_o *is the closure of the graph of the operator* (7.6) *in* $\ell_2 \times \ell_2$ *and the graph of* $A_o|_{\ell_2}$ *is the closure of the graph of the operator* (7.6) *in* $\operatorname{dom}(A_o^*)' \times \operatorname{dom}(A_o^*)'$.

Proof. Lemma 4.3.5(i) of [18] states that $\mathfrak{B}$ maps the set $W_0^{1,2}(\mathbb{R}_-; \mathcal{U})$ into $\operatorname{dom} A$, so the relation

$$\mathcal{Z} = V^* S^* \mathfrak{B} W_0^{1,2}(\mathbb{R}_-; \mathcal{U}) \subset V^* S^* (\operatorname{dom} A \cap \operatorname{ran} R) \subset \Sigma \ell_2$$

holds. This means that for arbitrary $z \in \mathcal{Z}$, we may write $z = V^* S^* y$ with $y \in \operatorname{dom} A \cap \operatorname{ran} R$. Then $S^* y \in \operatorname{ran} S^* R$, and with VV^* being the identity on this set, one gets

$$\pi_{(\ker \mathfrak{C})^\perp} y = S^{-*} S^* y = S^{-*} V V^* S^* y = S^{-*} V z = \pi_{(\ker S^*)^\perp} R U \Sigma^{-1} z$$

with (7.1). Recall that, according to Lemma 3.1, $\pi_{(\ker \mathfrak{C})^\perp}\mathfrak{A}|_{\overline{M}}$ is a semigroup whose generator $\widetilde{A}$ has the domain $\overline{M} \cap \pi_{(\ker \mathfrak{C})^\perp} \operatorname{dom} A$. Since $\pi_{(\ker \mathfrak{C})^\perp} y \in M \cap \pi_{(\ker \mathfrak{C})^\perp} \operatorname{dom} A$ is in this domain, the calculation

$$\begin{aligned}
&\lim_{t \downarrow 0} \frac{1}{t} \left(\mathcal{T}\mathfrak{A}(t)\mathcal{T}^+ z - z \right) \\
&= \lim_{t \downarrow 0} \frac{1}{t} \left(V^* S^* f A(t) \pi_{(\ker S^*)^\perp} R U \Sigma^{-1} z - z \right) \\
&= \lim_{t \downarrow 0} \frac{1}{t} \left(V^* S^* \mathfrak{A}(t) S^{-*} V (V^* S^*) y - (V^* S^*) y \right) \\
&= \lim_{t \downarrow 0} \frac{1}{t} V^* \left(S^* \pi_{(\ker S^*)^\perp} \mathfrak{A}(t) \pi_{(\ker S^*)^\perp} y - S^* \pi_{(\ker S^*)^\perp} y \right) \\
&= V^* S^* \lim_{t \downarrow 0} \frac{1}{t} \left(\pi_{(\ker S^*)^\perp} \mathfrak{A}(t) \pi_{(\ker S^*)^\perp} y - \pi_{(\ker S^*)^\perp} y \right) \\
&= V^* S^* \pi_{(\ker S^*)^\perp} \widetilde{A} \pi_{(\ker S^*)^\perp} y \\
&= V^* S^* \widetilde{A} \pi_{(\ker S^*)^\perp} R U \Sigma^{-1} z
\end{aligned}$$

shows $A_o z = V^* S^* \widetilde{A} S^{-*} V z$ and $V^* S^* (\operatorname{dom} A \cap \operatorname{ran} R) \subset \operatorname{dom} A_o$. By [5, Proposition II.1.7], $\mathcal{Z}$ is already a core for A_o if it is $\mathfrak{A}_o$ invariant and dense

in $\operatorname{dom} A_o$. It is indeed invariant: We can write any $z \in \mathcal{Z}$ as $z = V^*S^*\mathfrak{B}u$ with $u \in W_0^{1,2}(\mathbb{R}_-;\mathcal{U})$ and the equality

$$\begin{aligned}\mathfrak{A}_o z &= V^*S^*\mathfrak{A}(t)S^{-*}Vz = V^*S^*\mathfrak{A}(t)S^{-*}V(V^*S^*\mathfrak{B}u)\\ &= V^*S^*\mathfrak{A}(t)\pi_{(\ker S^*)^\perp}\mathfrak{B}u = V^*S^*\mathfrak{A}(t)\mathfrak{B}u = V^*S^*\mathfrak{B}\tau_-^t u\end{aligned}$$

holds. Now the left shift of u is obviously again in $W_0^{1,2}(\mathbb{R}_-;\mathcal{U})$ and the overall expression therefore in $\mathcal{Z}$. Regarding density, we have that the continuous mapping $V^*S^*\mathfrak{B}$ maps the dense subset $W_0^{1,2}(\mathbb{R}_-;\mathcal{U})$ of $L^2(\mathbb{R}_-;\mathcal{U})$ into a dense subset of its image $\operatorname{ran} V^*S^*\mathfrak{B}$, which is $\Sigma\ell_2$. Since this is dense in ℓ_2, we conclude that $\mathcal{Z}$ is dense in ℓ_2 and in particular in $\operatorname{dom} A_o$. For the assertion concerning the extended generator observe that any core of a semigroup generator is also a core for the extended generator, since the density with respect to the stronger graph norm implies density with respect to the weaker graph norm. □

Remark 7.1. If $\ker \mathfrak{C} = \{0\}$, i.e. the original system is observable, the projection $\pi_{(\ker S^*)^\perp}$ is just the identity and $\widetilde{A}$ may be replaced by A. In the non-observable case, one might be tempted to omit the projection in the expression $V^*S^*A\pi_{(\ker S^*)^\perp}RU\Sigma^{-1}$ as well, since A maps $\operatorname{dom} A \cap \ker S^*$ into $\ker S^*$ anyway. However, this is not allowed because for arbitrary $z \in \mathcal{Z}$, $RU\Sigma^{-1}z$ will in general not be in the domain of A, even though the projected vector $\pi_{(\ker S^*)^\perp}RU\Sigma^{-1}z$ lies in $\pi_{(\ker S^*)^\perp}\operatorname{dom} A$.

Proof of Theorem 5.3. We start with the poof of *c)*. A simple calculation shows that the adjoint $(\mathcal{T}|_{\overline{M}})^*$ of $\mathcal{T}|_{\overline{M}} : \overline{M} \to \ell_2$ equals $\pi_{\overline{M}}SV$. In order to show that $(\mathcal{T}|_{\overline{M}})^*$ maps $\operatorname{dom} A_o^*$ into $\operatorname{dom} \widetilde{A}^*$, we prove the following three auxiliary statements:

(i) For all $(x_n) \in \ell_2$ holds $SV(x_n) = \mathfrak{C}^*\widetilde{V}(x_n)$: Due to continuity, the equality

$$Sx = SS^*S^{-*}x = \mathfrak{C}^*\mathfrak{C}S^{-*}x = \mathfrak{C}^*\mathfrak{V}x,$$

which is true for all $x \in \operatorname{ran} S^*R$, must hold on $\overline{\operatorname{ran} S^*R} = \operatorname{ran} V$ as well, and the assertion follows using $\widetilde{V} = \mathfrak{V}V$.

(ii) The operator $\mathfrak{C}^*$ maps $W_0^{1,2}(\mathbb{R}_+;\mathcal{Y})$ into $\operatorname{dom} A^*$: This follows because $\mathfrak{C}^*\Gamma$ is the input operator of the adjoint system [18, Section 6.2] when Γ denotes the reflection operator from (2.4), and therefore, $\mathfrak{C}^*\Gamma$ maps $W_0^{1,2}(\mathbb{R}_-;\mathcal{Y})$ into $\operatorname{dom} A^*$, according to Lemma 4.3.5(i) of [18], which was already used in the proof of Proposition 7.3.

(iii) The last assertion is that $\pi_{\overline{M}}\mathfrak{C}^* = \pi_{\overline{M}}\mathfrak{C}^*\pi_{\overline{\operatorname{ran}\mathfrak{H}}}$: If we take an arbitrary $y \in L^2(\mathbb{R}_+;\mathcal{Y})$, then $(\overline{\operatorname{ran}\mathfrak{H}})^\perp = \ker \mathfrak{B}^*\mathfrak{C}^*$ shows that

$$\widetilde{y} := \mathfrak{C}^*\pi_{\overline{\operatorname{ran}\mathfrak{H}}^\perp}y \quad \in \quad \operatorname{ran}\mathfrak{C}^* \cap \ker\mathfrak{B}^* \quad \subset \quad (\ker\mathfrak{C})^\perp \cap (\operatorname{ran}\mathfrak{B})^\perp.$$

Hence, taking the scalar product with any $x \in M$, which must be of the form $x = \pi_{(\ker\mathfrak{C})^\perp}b$ for some $b \in \operatorname{ran}\mathfrak{B}$ yields

$$\langle x, \widetilde{y}\rangle_X = \langle b - \pi_{\ker\mathfrak{C}}b, \widetilde{y}\rangle_X = \langle b, \widetilde{y}\rangle_X - \langle \pi_{\ker\mathfrak{C}}b, \widetilde{y}\rangle_X = 0$$

It follows $\widetilde{y} \in \overline{M}^{\perp}$, and therefore

$$\begin{aligned}\pi_{\overline{M}}\mathfrak{C}^* y &= \pi_{\overline{M}}\mathfrak{C}^*\pi_{\overline{\operatorname{ran}\mathfrak{H}}} y + \pi_{\overline{M}}\mathfrak{C}^*\pi_{(\overline{\operatorname{ran}\mathfrak{H}})^{\perp}} y \\ &= \pi_{\overline{M}}\mathfrak{C}^*\pi_{\overline{\operatorname{ran}\mathfrak{H}}} y + \pi_{\overline{M}}\widetilde{y} \\ &= \pi_{\overline{M}}\mathfrak{C}^*\pi_{\overline{\operatorname{ran}\mathfrak{H}}} y,\end{aligned}$$

which is what we wanted to show.

In order to prove our original claim, we pick $(x_n) \in \operatorname{dom} A_o^*$. Because $\widetilde{V}$ was the similarity transformation between (5.5) and the output-normalized shift realization, we have $\widetilde{V} \operatorname{dom} A_o^* = \pi_{\overline{\operatorname{ran}\mathfrak{C}\mathfrak{B}}} W_0^{1,2}(\mathbb{R}_+;\mathcal{Y})$, see (7.3a). Hence, $\widetilde{V}(x_n) = \pi_{\overline{\operatorname{ran}\mathfrak{C}\mathfrak{B}}} y$ for some $y \in W_0^{1,2}(\mathbb{R}_+;\mathcal{Y})$, and with (i) and (iii) we get

$$(\mathcal{T}|_{\overline{M}})^*(x_n) = \pi_{\overline{M}} S V(x_n) = \pi_{\overline{M}}\mathfrak{C}^*\widetilde{V}(x_n) = \pi_{\overline{M}}\mathfrak{C}^*\pi_{\overline{\operatorname{ran}\mathfrak{C}\mathfrak{B}}} y = \pi_{\overline{M}}\mathfrak{C}^* y.$$

Now, because of (ii), the latter is an element of $\pi_{\overline{M}}(\operatorname{dom} A^* \cap (\ker\mathfrak{C})^{\perp})$, which was shown to be $\operatorname{dom}\widetilde{A}^*$ in Lemma 3.1. Finally, Lemma 1.2 implies that (5.8) is an extension of $\mathcal{T}|_{\overline{M}}$ as claimed in *c)*.

Observe that, on the set $\mathcal{Z}$, the operators $A_o|_{\ell_2}$ and $\mathcal{T}_{-1}\widetilde{A}\pi_{(\ker S^*)^{\perp}}\mathcal{T}^+$ reduce to their unextended versions and therefore coincide according Proposition 7.3. Since $\mathcal{Z}$ is a core of the closed operator $A_o|_{\ell_2}$, whose domain contains $\Sigma\ell_2$, this shows that $\mathcal{T}_{-1}\widetilde{A}\pi_{(\ker S^*)^{\perp}}\mathcal{T}^+$ is closable and its closure is $A_o|_{\ell_2}$. In particular, both operators coincide on the larger set $\Sigma\ell_2$, hence the assertions (5.6) and (5.9) are true. We make use of this fact to determine the control operator via (2.7). For any $u \in \mathcal{U}$ and λ in $\rho(A_o) \cap \rho(\widetilde{A})$ it can be calculated by

$$\begin{aligned}B_o u &= (\lambda - A_o|_{\ell_2})\mathfrak{B}_o e_\lambda u = \left(\lambda - \mathcal{T}_{-1}\widetilde{A}\pi_{(\ker S^*)^{\perp}}\mathcal{T}^+\right)\mathcal{T}\pi_{(\ker S^*)^{\perp}}\mathfrak{B} e_\lambda u \\ &= \mathcal{T}_{-1}(\lambda - \widetilde{A})\pi_{(\ker S^*)^{\perp}}\mathfrak{B} e_\lambda u = \mathcal{T}_{-1}(\lambda - \widetilde{A})\widetilde{\mathfrak{B}} e_\lambda u = \mathcal{T}_{-1}\widetilde{B} u.\end{aligned}$$

Here, we have used that $\mathcal{T}$ maps M into $\Sigma\ell_2$ and $\pi_{(\ker S^*)^{\perp}}\mathcal{T}^+\mathcal{T}$ is the identity on M. Now for the output operator C_o: We take an element $z \in \Sigma\ell_2$ such that $z = \mathcal{T}x$ for some $x \in \operatorname{dom} C$. Then in general $\mathcal{T}^+ z \neq x$, and the first thing we have to check is that $\mathcal{T}^+ z$ is in the domain of C. An immediate consequence of the definition of $\operatorname{dom} C$ is that $\ker S^* = \ker\mathfrak{C} \subset (\operatorname{dom} C \cap \ker C)$. Since $\operatorname{dom} C$ is a linear space, we deduce

$$\pi_{(\ker\mathfrak{C})^{\perp}}\mathcal{T}^+ z = \pi_{(\ker\mathfrak{C})^{\perp}} x = x - \pi_{\ker\mathfrak{C}} x \in \operatorname{dom} C,$$

and with this we get indeed

$$\mathcal{T}^+ z = \pi_{(\ker\mathfrak{C})^{\perp}}\mathcal{T}^+ z + \pi_{\ker\mathfrak{C}}\mathcal{T}^+ z \in \operatorname{dom} C.$$

Hence,

$$C_o z = \lim_{t\to 0}\frac{1}{t}\int_0^t (\mathfrak{C} S^{-*} V z)(\tau)\,d\tau = C S^{-*} V z = C\pi_{\ker\mathfrak{C}}\mathcal{T}^+ z = C\mathcal{T}^+ z$$

and $\mathcal{T}\operatorname{dom} C \cap \Sigma\ell_2 \subset \operatorname{dom} C_o$. □

8. Proof of Theorem 5.4 and 5.5

The idea behind the proof of Theorem 5.4 is to obtain the balanced system by interpolating between the output normalized realization (5.5) of $\mathfrak{D}$ on $(\ell_2, \langle\cdot,\cdot\rangle_{\ell_2})$ and its restriction to $(\Sigma\ell_2, \langle\cdot,\cdot\rangle_\Sigma)$. This restriction is an input normalized realization of $\mathfrak{D}$, and it is well-posed because it is unitary equivalent to the input normalized system (4.1). So an important ingredient for the proof is the following auxiliary result about well-posedness of an interpolated system.

Lemma 8.1. *Let $\underline{X}$, X and $\overline{X}$ be Hilbert spaces with $\underline{X} \hookrightarrow X \hookrightarrow \overline{X}$. Assume that there exists a positive operator $\Sigma \in \mathcal{B}(\overline{X})$ such that $X = \operatorname{ran}\Sigma^{1/2}$, $\underline{X} = \operatorname{ran}\Sigma$ and*

$$\langle x, y\rangle_{\overline{X}} = \langle \Sigma^{1/2}x, \Sigma^{1/2}y\rangle_X = \langle \Sigma x, \Sigma y\rangle_{\underline{X}} \quad \forall x, y \in \overline{X}.$$

Let $(\overline{\mathfrak{A}}, \underline{\mathfrak{B}}, \overline{\mathfrak{C}}, \mathfrak{D})$ and $(\underline{\mathfrak{A}}, \underline{\mathfrak{B}}, \underline{\mathfrak{C}}, \mathfrak{D})$ be two bounded well-posed linear systems on the Hilbert spaces $(\mathcal{U}, \overline{X}, \mathcal{Y})$ and $(\mathcal{U}, \underline{X}, \mathcal{Y})$ respectively, with the same input map $\underline{\mathfrak{B}}$, the same input/output map $\mathfrak{D}$ and $\underline{\mathfrak{A}} = \overline{\mathfrak{A}}|_{\underline{X}}$, $\underline{\mathfrak{C}} = \overline{\mathfrak{C}}|_{\underline{X}}$. Then X is invariant under $\overline{\mathfrak{A}}$ and $(\overline{\mathfrak{A}}|_X, \underline{\mathfrak{B}}, \overline{\mathfrak{C}}|_X, \mathfrak{D})$ is a bounded well-posed linear system on $(\mathcal{U}, X, \mathcal{Y})$. Moreover the domain of the generator A of $\mathfrak{A}$ is the part of $\overline{\mathfrak{A}}$ in X and the domain of $\underline{A}$ is a core for A.

Proof. The claim about well-posedness is a special case of Lemma 9.5.7 in [18]. That the generator of a semigroup restricted to an invariant subspace is given by the part of the generator in the subspace, is Lemma 1.1. To see that $\operatorname{dom}\underline{A}$ is a core, it suffices by [5, Proposition II.1.7] to see that it is invariant under $\mathfrak{A}$ and a dense subset of $\operatorname{dom} A$. The latter is true because of $\underline{X} \hookrightarrow X$. □

Remark 8.1. As an immediate consequence of Lemma 7.1 and the fact that $\Sigma^{1/2} : \ell_2 \to \Sigma^{1/2}\ell_2$ is an isomorphism the mapping

$$T|_M : M \subset X \to \Sigma^{1/2}\ell_2, \quad T := \Sigma^{-1/2}V^*S^*.$$

is an isomorphism with inverse

$$T^+ : \Sigma^{1/2}\ell_2 \subset \ell_2 \to M, \quad \pi_{(\ker S^*)^\perp}T^+ := \pi_{(\ker S^*)^\perp}RU\Sigma^{-1/2}.$$

Lemma 8.2. *Under the conditions of Theorem 5.4 the following holds: The output normalized system (5.5), restricted to $\Sigma\ell_2$,*

$$(\underline{\mathfrak{A}}, \underline{\mathfrak{B}}, \underline{\mathfrak{C}}, \mathfrak{D}) := (\mathfrak{A}_o|_{\Sigma\ell_2}, \mathfrak{B}_o, \mathfrak{C}_o|_{\Sigma\ell_2}, \mathfrak{D}) \tag{8.1}$$

defines an input-normalized, bounded well-posed linear system on $(\Sigma\ell_2, \|\cdot\|_\Sigma)$. The generator $\underline{A}$ of $\underline{\mathfrak{A}}$ satisfies

$$\begin{aligned} \operatorname{dom}\underline{A} &= \mathcal{Z} = \{x \in \operatorname{dom} A_o \cap \Sigma\ell_2 : A_o x \in \Sigma\ell_2\}, \\ \underline{A} &: \operatorname{dom}\underline{A} \subset \Sigma\ell_2 \to \Sigma\ell_2, \quad \underline{A}x = A_o x \quad \forall x \in \operatorname{dom}\underline{A}, \end{aligned} \tag{8.2}$$

with $\mathcal{Z}$ as in Theorem 5.3 a).

Proof. Similar to the proof of Theorem 5.2, we claim that

$$\mathfrak{U}U\Sigma^{-1} : \Sigma\ell_2 \to (\ker \mathfrak{H})^\perp$$

with inverse $\Sigma\mathfrak{U}^*U^*$ is a unitary similarity transformation between the system (8.1) and the input normalized shift realization (4.1) on $(\ker \mathfrak{H})^\perp$. We know that $U \in \mathcal{B}(\ell_2; (\ker S^*R)^\perp)$ and $\mathfrak{U} \in \mathcal{B}((\ker S^*R)^\perp); (\ker \mathfrak{H})^\perp)$ are unitary (Lemma 6.1). Due to the scalar product (1.3) used on $\Sigma\ell_2$, the operator $\Sigma \in \mathcal{B}(\ell_2; \Sigma\ell_2)$ is unitary as well. So it only remains to show that (8.1) is related to (4.1) via these transformations. In Lemma 6.1 we proved that $R^{-1}\mathfrak{B} : \ker \mathfrak{B}^\perp \to \ker R^\perp$ maps $(\ker \mathfrak{H})^\perp$ into $(\ker S^*R)^\perp$ and called the restriction $\mathfrak{U}^*$. Using the readily verified fact that $R^{-1}\mathfrak{B}$ also maps $\ker \mathfrak{H}$ into $\ker S^*R$, we therefore have

$$\pi_{(\ker S^*R)^\perp}(R^{-1}\mathfrak{B}) = \pi_{(\ker S^*R)^\perp}(R^{-1}\mathfrak{B})\pi_{(\ker \mathfrak{H})^\perp} = \mathfrak{U}^*\pi_{(\ker \mathfrak{H})^\perp},$$

and consequently for all $x \in \operatorname{ran} \mathfrak{B}$

$$\begin{aligned} V^*S^*x &= V^*S^*RR^{-1}\mathfrak{B}\mathfrak{B}^{-1}x = V^*S^*R\pi_{(\ker S^*R)^\perp}(R^{-1}\mathfrak{B})\mathfrak{B}^{-1}x \\ &= \Sigma U^*\mathfrak{U}^*\pi_{(\ker \mathfrak{C}\mathfrak{B})^\perp}\mathfrak{B}^{-1}x. \end{aligned} \tag{8.3}$$

Using this and the $\mathfrak{A}$-invariance of $\operatorname{ran} R = \operatorname{ran} \mathfrak{B}$, we have

$$\begin{aligned} \mathfrak{A}_o(t)|_{\Sigma\ell_2} &= V^*S^*\mathfrak{A}(t)RU\Sigma^{-1} \\ &= \Sigma U^*\mathfrak{U}^*\pi_{(\ker \mathfrak{C}\mathfrak{B})^\perp}\mathfrak{B}^{-1}\mathfrak{A}(t)\mathfrak{B}\mathfrak{U}U\Sigma^{-1} \\ &= \Sigma U^*\mathfrak{U}^*\pi_{(\ker \mathfrak{C}\mathfrak{B})^\perp}\tau_-^t|_{(\ker \mathfrak{C}\mathfrak{B})^\perp}\mathfrak{U}U\Sigma^{-1} \end{aligned}$$

This shows that $\pi_{(\ker S^*R)^\perp}R^{-1}\mathfrak{A}R$ is unitarily similar to the strongly continuous semigroup of the shift realization on $(\ker \mathfrak{C}\mathfrak{B})^\perp$. For the input operators, equation (8.3) immediately gives the asserted formula

$$V^*S^*\mathfrak{B} = \Sigma U^*\mathfrak{U}^*\pi_{(\ker \mathfrak{C}\mathfrak{B})^\perp}\pi_{(\ker \mathfrak{B})^\perp} = \Sigma U^*\mathfrak{U}^*\pi_{(\ker \mathfrak{C}\mathfrak{B})^\perp},$$

and finally, the output operator $\underline{\mathfrak{C}}$ equals

$$\mathfrak{C}RU\Sigma^{-1} = \mathfrak{C}\mathfrak{B}(\mathfrak{B}^{-1}R)U\Sigma^{-1} = \mathfrak{H}|_{(\ker \mathfrak{H})^\perp}\mathfrak{U}U\Sigma^{-1},$$

which is the output operator the exactly controllable shift realization. So we have shown that system (8.1) is but a similarity transformation of the system (4.1). Therefore, well-posedness follows from Lemma 2.1 and moreover, the unitary transformations keep the system input normalized.

The domain of $\underline{A}$ is given by the transformations $\Sigma\mathfrak{U}^*U^*$ applied to the domain of the exactly controllable shift realization. With (8.3) this becomes

$$\operatorname{dom} \underline{A} = \Sigma\mathfrak{U}^*U^*\pi_{(\ker \mathfrak{H})^\perp}W_0^{1,2}(\mathbb{R}_-; \mathcal{U}) = V^*S^*\mathfrak{B}W_0^{1,2}(\mathbb{R}_-, \mathcal{U}) = \mathcal{Z}.$$

On the other hand, we know that $\underline{\mathfrak{A}}$ is the restriction of $\mathfrak{A}_o$ and strongly continuous with respect to $\|\cdot\|_{\Sigma\ell_2}$. Hence, Lemma 1.1 tells us that the generator $\underline{A}$ is must be the part of A_o in $\Sigma\ell_2$, which is by definition the first term in (8.2). □

Remark 8.2. a) We point out that one of the essential properties of the space $\mathcal{Z}$ is that $x \in \mathfrak{B}W_0^{1,2}(\mathbb{R}_-;\mathcal{U})$ does not only imply $x \in \operatorname{dom} A \cap \operatorname{ran}\mathfrak{B}$, but also $Ax \in \operatorname{ran}\mathfrak{B}$. This explains the fact that V^*S^*Ax is again an element of $\Sigma\ell_2$ and thus the relation (8.2).

b) A further similarity transformation with Σ yields an input normalized system on the state space ℓ_2. This gives a completely analogous result to Theorem 5.2 with output normalization replaced by input normalization. For the upcoming interpolation step however, the present system with state space $\Sigma\ell_2$ is more convenient.

Proof of Theorem 5.4. We apply the interpolation Lemma 8.1 to the output normalized system (5.5) on ℓ_2 and its restriction (8.1) to $\Sigma\ell_2$. This guarantees the well-posedness of the system

$$\left(\widehat{\mathfrak{A}},\ \widehat{\mathfrak{B}},\ \widehat{\mathfrak{C}},\ \mathfrak{D}\right) := \left(\mathfrak{A}_o|_{\Sigma^{1/2}\ell_2},\ \mathfrak{B}_o,\ \mathfrak{C}_o|_{\Sigma^{1/2}\ell_2},\ \mathfrak{D}\right) \tag{8.4}$$

on the interpolated state space $\Sigma^{1/2}\ell_2$. In particular, $\Sigma^{1/2}$ is invariant under $\mathfrak{A}_o$ and $\mathfrak{A}_o|_{\Sigma^{1/2}\ell_2}$ is strongly continuous with respect to $\|\cdot\|_{\Sigma^{1/2}\ell_2}$.

In order to determine the Gramians of this system, we calculate the adjoints with respect to $\langle\cdot,\cdot\rangle_{\Sigma^{1/2}\ell_2}$. For all $y \in L^2(\mathbb{R}_+;\mathcal{Y})$ and $u \in L^2(\mathbb{R}_-;\mathcal{U})$ we have

$$\begin{aligned}\langle \mathfrak{C}_o x, y\rangle_{L^2(\mathbb{R}_+;\mathcal{Y})} &= \langle x, \mathfrak{C}_o^* y\rangle_{\ell_2}\\ &= \langle \Sigma^{-1/2}x, \Sigma^{-1/2}\Sigma\mathfrak{C}_o^* y\rangle_{\ell_2}\\ &= \langle x, \Sigma\mathfrak{C}_o^* y\rangle_{\Sigma^{1/2}\ell_2} \qquad \forall x \in \Sigma^{1/2}\ell_2 \qquad \text{and}\end{aligned}$$

$$\begin{aligned}\langle \mathfrak{B}_o u, x\rangle_{\Sigma^{1/2}\ell_2} &= \langle \Sigma^{-1/2}\mathfrak{B}_o u, \Sigma^{-1/2}x\rangle_{\ell_2}\\ &= \langle u, \mathfrak{B}_o^*\Sigma^{-1}x\rangle_{L^2(\mathbb{R}_-;\mathcal{U})} \qquad \forall x \in \Sigma\ell_2.\end{aligned}$$

Thus, the observability and controllability Gramians with respect to the scalar product $\langle\cdot,\cdot\rangle_{\Sigma^{1/2}\ell_2}$ are given by

$$\widehat{\mathfrak{C}}^*\widehat{\mathfrak{C}} = \Sigma\mathfrak{C}_o^*\mathfrak{C}_o = \Sigma\,\mathrm{id}_{\Sigma^{1/2}} \qquad \text{and}$$

$$\begin{aligned}\widehat{\mathfrak{B}}\widehat{\mathfrak{B}}^*x &= \mathfrak{B}_o\mathfrak{B}_o^*\Sigma^{-1}x = V^*S^*\mathfrak{B}\mathfrak{B}^*SV\Sigma^{-1}x = V^*S^*RR^*SV\Sigma^{-1}x\\ &= V^*S^*R\pi_{(\ker S^*R)^\perp}R^*SV\Sigma^{-1}x = V^*S^*RUU^*R^*SV\Sigma^{-1}x\\ &= \Sigma x \quad \forall x \in \Sigma\ell_2,\end{aligned}$$

where the last equation can be extended to the whole space $\Sigma^{1/2}\ell_2$, because both of the operators $\widehat{\mathfrak{B}}\widehat{\mathfrak{B}}^*$ and Σ are in $\mathcal{B}(\Sigma^{1/2}\ell_2)$.

The last step is just to transfer the system to the favored state space ℓ_2 via another unitary transformation $\Sigma^{-1/2} : \Sigma^{1/2}\ell_2 \to \ell_2$. The result of this is the system

$$\begin{aligned}&\left(\Sigma^{-1/2}\widehat{\mathfrak{A}}\Sigma^{1/2},\quad \Sigma^{-1/2}\widehat{\mathfrak{B}},\quad \widehat{\mathfrak{C}}\Sigma^{1/2},\quad \mathfrak{D}\right)\\ &= \left(\Sigma^{-1/2}\overline{V^*S^*\mathfrak{A}RU\Sigma^{-1}}\Sigma^{1/2},\quad \Sigma^{-1/2}V^*S^*\mathfrak{B},\quad \overline{\mathfrak{C}RU\Sigma^{-1}}\Sigma^{1/2},\quad \mathfrak{D}\right)\end{aligned} \tag{8.5}$$

on ℓ_2. Since we are transforming unitarily with respect to the scalar products $\langle\cdot,\cdot\rangle_{\Sigma^{1/2}\ell_2}$ and $\langle\cdot,\cdot\rangle_{\ell_2}$, the Gramians do not change and the resulting system is still balanced. In order to complete the proof, it suffices to check that the operators defined in (5.12) and (8.5) are the same. For $\mathfrak{B}_o$ and $\mathfrak{D}$ there is nothing to prove. For $\mathfrak{A}_o(t)$ and $\mathfrak{C}_o$ it follows since all the operators are bounded with respect to the ℓ_2-norm and coincide on the dense subset $\Sigma\ell_2$ of ℓ_2. □

Proof of Theorem 5.5 . Lemma 8.1 also tells us that $\mathcal{Z} = \operatorname{dom}\underline{A}$ is a core for $\widehat{A}$ and that the domain of $\widehat{A}$ is the part of A_o in $\Sigma^{1/2}\ell_2$. This means in particular

$$\operatorname{dom}\widehat{A} = \left\{x \in \Sigma^{1/2} \cap \operatorname{dom} A_o : A_o x \in \Sigma^{1/2}\ell_2\right\},$$
$$\widehat{A}z = A_o z = V^* S^* \widetilde{A}\pi_{(\ker S^*)^\perp} RU\Sigma^{-1}x \quad \forall z \in \mathcal{Z}.$$

Since the semigroups $\mathfrak{A}_b$ and $\widehat{\mathfrak{A}}$ in the proof of Theorem 5.3 are unitarily similar via the transformation $\Sigma^{-1/2} \in \mathcal{B}(\Sigma^{1/2}\ell_2;\ell_2)$, the same is true for their generators. So $Z = \Sigma^{-1/2}\mathcal{Z}$ must be a core for A_b and

$$\operatorname{dom} A_b = \Sigma^{-1/2}\operatorname{dom}\widehat{A} = \Sigma^{-1/2}\left\{x \in \Sigma^{1/2} \cap \operatorname{dom} A_o : A_o x \in \Sigma^{1/2}\ell_2\right\}$$
$$A_b z = \Sigma^{-1/2}\widehat{A}\Sigma^{1/2} = \Sigma^{-1/2}V^* S^* \widetilde{A}\pi_{(\ker S^*)^\perp} RU\Sigma^{-1/2}z \quad \forall z \in Z.$$

The claim about the representation of C_b follows from the fact that $\mathfrak{C}_b x = \mathfrak{C}_o\Sigma^{1/2}x$ for all $x \in \Sigma^{-1/2}(V^*S^*\operatorname{dom} C \cap \Sigma\ell_2)$ and (5.7) via the definition of the Lebesgue extension in (2.6). So we have proved *a)* and *b)*.

We do not determine the domain of the adjoint A_b^* exactly, but we will prove that $\widetilde{Z} := \Sigma^{1/2}\operatorname{dom} A_o^*$ is a core for A_b^*. Take $y \in \widetilde{Z}$ and $x \in \operatorname{dom} A_b \subset \Sigma^{-1/2}\operatorname{dom} A_o$. Then the equation

$$\langle A_b x, y\rangle_{\ell_2} = \langle \Sigma^{-1/2}A_o\Sigma^{1/2}x, y\rangle_{\ell_2} = \langle x, \Sigma^{1/2}A_o^*\Sigma^{-1/2}y\rangle_{\ell_2}$$

shows, since the right hand side is continuous in x, that $y \in \operatorname{dom} A_b^*$ and

$$A_b^* y = \Sigma^{1/2}A_o^*\Sigma^{-1/2}y \qquad \forall y \in \widetilde{Z}. \tag{8.6}$$

So we have shown $\widetilde{Z} \subset \operatorname{dom} A_b^*$. We now prove that $\widetilde{Z}$ is dense in ℓ_2 and $\mathfrak{A}_b^*$-invariant. The continuity of $\Sigma^{1/2} \in \mathcal{B}(\ell_2)$ implies that $\Sigma^{1/2}\operatorname{dom} A_o^*$ is dense in $\Sigma^{1/2}\ell_2$ with respect to the topology of ℓ_2. Because $\Sigma^{1/2}\ell_2$ itself is dense in ℓ_2, it follows that $\widetilde{Z}$ is dense in ℓ_2. Furthermore, for $x \in \ell_2$ and $y \in \widetilde{Z} \subset \Sigma^{1/2}\ell_2$, the equation

$$\langle \mathfrak{A}_b(t)x, y\rangle_{\ell_2} = \langle \Sigma^{-1/2}\mathfrak{A}_o(t)\Sigma^{1/2}x, y\rangle_{\ell_2} = \langle x, \Sigma^{1/2}\mathfrak{A}_o^*(t)\Sigma^{-1/2}y\rangle_{\ell_2}$$

shows $\mathfrak{A}_b^*(t)y = \Sigma^{1/2}\mathfrak{A}_o^*(t)\Sigma^{-1/2}y$. This representation together with the definition of $\widetilde{Z}$ show the $\mathfrak{A}_b$-invariance of $\widetilde{Z}$, since the $\mathfrak{A}_o(t)$ maps $\operatorname{dom} A_o^*$ into itself. So altogether $\widetilde{Z}$ must be a core of A_b^*. To complete the proof of *c)*, we observe that $(T|_{\overline{M}})^* = (\mathcal{T}|_{\overline{M}})^*\Sigma^{-1/2}$ and therefore

$$(T|_{\overline{M}})^*\widetilde{Z} = (\mathcal{T}|_{\overline{M}})^*\Sigma^{-1/2}\Sigma^{1/2}\operatorname{dom} A_o^* = (\mathcal{T}|_{\overline{M}})^*\operatorname{dom} A_o^*.$$

The latter is by Theorem 5.3 *c)* a subset of $\operatorname{dom}\widetilde{A}^*$.

Finally, we prove *d)*. Choose λ in the resolvent sets of A_b and A_o, and take y still in $\widetilde{Z}$ and $u \in \mathcal{U}$. Knowing from *a)* that $y \in \operatorname{dom} A_b^*$ and using (8.6) we have

$$\begin{aligned}
\langle B_b u, y\rangle_{(\operatorname{dom} A_b^*)',\operatorname{dom} A_b^*} &= \langle (\lambda - A_b|_{\ell_2})\mathfrak{B}_b e_\lambda u, y\rangle_{(\operatorname{dom} A_b^*)',\operatorname{dom} A_b^*} \\
&= \langle \mathfrak{B}_b e_\lambda u, (\overline{\lambda} - A_b^*)y\rangle_{\ell_2} \\
&= \langle \Sigma^{-1/2}\mathfrak{B}_o e_\lambda u, \Sigma^{1/2}(\overline{\lambda} - A_o^*)\Sigma^{-1/2}y\rangle_{\ell_2} \\
&= \langle \mathfrak{B}_o e_\lambda u, (\overline{\lambda} - A_o^*)\Sigma^{-1/2}y\rangle_{\ell_2} \\
&= \langle (\lambda - A_o|_{\ell_2})\mathfrak{B}_o u, \Sigma^{-1/2}y\rangle_{(\operatorname{dom} A_o^*)',\operatorname{dom} A_o^*} \\
&= \langle B_o u, \Sigma^{-1/2}y\rangle_{(\operatorname{dom} A_o^*)',\operatorname{dom} A_o^*} \\
&= \langle \widetilde{B}u, (\mathcal{T}|_{\overline{M}})^* y\rangle_{(\operatorname{dom} \widetilde{A}^*)',\operatorname{dom} \widetilde{A}^*}. \qquad (8.7)
\end{aligned}$$

The functional $B_b u \in (\operatorname{dom} A_b^*)'$ is obtained by continuous extension of this expression to all $y \in \operatorname{dom} A_b^*$, because the core $\widetilde{Z}$ is dense in $\operatorname{dom} A_b^*$ with respect to the graph norm of A_b^*. □

Remark 8.3. The generator A_b of $\mathfrak{A}_b$ is also equal to

$$\operatorname{dom} A_b = \Sigma^{-1/2}\left\{(x_n) \in \Sigma^{1/2}\ell_2 \;\middle|\; \begin{array}{l} \widetilde{V}(x_n) \in W^{1,2}(\mathbb{R}_+;\mathcal{Y}) \quad \text{and} \\ \widetilde{V}^*(\frac{d}{d\xi})\widetilde{V}(x_n) \in \Sigma^{1/2}\ell_2 \end{array}\right\},$$

$$A_b x = \Sigma^{-1/2}\widetilde{V}^*(\tfrac{d}{d\xi})\widetilde{V}\Sigma^{1/2}x.$$

9. Proof of Theorem 5.6 and 5.7

We want to give a short explanation as to why it is necessary to define the spaces Z_b, respectively Z_o, in Theorem 5.6 and 5.7. The easiest way to see this is in the latter theorem, when we try to truncate the functional $A_o|_{\ell_2}e_j \in (\operatorname{dom} A_o^*)'$ from (7.2b). This functional is, by Lemma 1.2, defined through the adjoint (7.3b), and can, by partial integration, be shown to equal

$$\langle A_o|_{\ell_2}e_j, (x_n)\rangle_{\operatorname{dom}(A_o^*)',\operatorname{dom}(A_o^*)} = \int_0^\infty \tfrac{d}{d\xi}\widetilde{v}_j(\xi)\sum_{n=1}^\infty \widetilde{v}_n(\xi)x_n d\xi,$$

with $\frac{d}{d\xi}\widetilde{v}_j \in L^1(\mathbb{R}_+;\mathcal{Y})$. This representation is valid for all (x_n) in the domain of A_o^*, which means $\sum_{n=1}^\infty \widetilde{v}_n x_n \in W_0^{1,2}(\mathbb{R}_+;\mathcal{Y})$. At a first glance, it may seem straight forward to truncate this expression by defining

$$\langle (A_o|_{\ell_2}e_j)_r, x\rangle_{\mathbb{C}^r} := \int_0^\infty \tfrac{d}{d\xi}\widetilde{v}_j(\xi)\sum_{n=1}^r \widetilde{v}_n(\xi)x_n d\xi$$

for $x := [x_1,\dots,x_r]^\top \in \mathbb{C}^r$. To do this properly, however, we need to extend the functional $A_o|_{\ell_2}e_j$ to $\{(x_n) \in \ell_2 : \sum_{n=1}^\infty \widetilde{v}_n x_n \in W^{1,1}(\mathbb{R}_+;\mathcal{Y})\}$, because $\sum_{n=1}^r \widetilde{v}_n x_n$ is an element of $W^{1,1}(\mathbb{R}_+;\mathcal{Y})$ and not of $W_0^{1,2}(\mathbb{R}_+;\mathcal{Y})$.

Since $W_0^{1,1}(\mathbb{R}_+;\mathcal{Y})$ is not dense in $W^{1,1}(\mathbb{R}_+;\mathcal{Y})$, this extension can not simply be obtained from continuity. Instead, it is constructed as follows: First, the subspace of all functionals in $(W_0^{1,1}(\mathbb{R}_+;\mathcal{Y}))'$ that can be represented by an L^1-function is identified with the actual space $L^1(\mathbb{R}_+;\mathcal{Y})$, and then it is embedded into the dual space of $W^{1,1}(\mathbb{R}_+;\mathcal{Y})$. For the operator B_o, which experiences the same difficulties only with $\frac{d}{d\xi}\widetilde{v}_j$ replaced by hu, one can either proceed in the same way or, more elegantly, by using the Lebesgue extension of its adjoint.

We only execute the proof for the slightly more difficult Theorem 5.6, because the one for Theorem 5.6 is analogous up to some simplifications.

Proof of Theorem 5.6. We show that the space

$$Z_b := \left\{ (z_n) \in \Sigma^{1/2}\ell_2 \,\middle|\, \widetilde{V}\Sigma^{-1/2}(z_n) \in W^{1,1}(\mathbb{R}_+;\mathcal{Y}) \right\}$$

with the norm $\|(z_n)\|_{Z_b} := \|\widetilde{V}\Sigma^{-1/2}(z_n)\|_{W^{1,1}(\mathbb{R}_+;\mathcal{Y})}$ has the asserted properties.

Because the Schmidt vectors $\widetilde{v}_i$ are in $W^{1,1}(\mathbb{R}_+;\mathcal{Y})$ by [8], the vector $\widetilde{V}\Sigma^{-1/2}e_i = 1/(\sqrt{\sigma_i})\widetilde{v}_i$ is also in this space. Hence, $e_i \in Z_b$ and *a)* is true.

We claim that the space

$$\widetilde{Z}_b := \left\{ f \in (\operatorname{dom} A_b^*)' \,\middle|\, \begin{array}{l} \exists \widetilde{f} \in L^1(\mathbb{R}_+;\mathcal{Y}) \quad \forall (x_n) \in \Sigma^{1/2}\operatorname{dom} A_o^* : \\ \langle f, (x_n)\rangle = \displaystyle\int_0^\infty \left\langle \widetilde{f}(\xi), y(\xi)\right\rangle_{\mathcal{Y}} d\xi \quad \text{for some} \\ y \in W_0^{1,2}(\mathbb{R}_+;\mathcal{Y}) \text{ with } \pi_{\overline{\operatorname{ran}\mathfrak{H}}} y = \widetilde{V}\Sigma^{-1/2}(x_n). \end{array} \right\}$$

with norm $\|f\|_{\widetilde{Z}_b} := \|\widetilde{f}\|_{L^1(\mathbb{R}_+;\mathcal{Y})}$ is continuously embedded into Z_b' via the injection

$$\iota : \widetilde{Z}_b \to Z_b', \qquad \langle \iota f, (z_n)\rangle := \int_0^\infty \left\langle \widetilde{f}(\xi), \sum_{n=1}^\infty \frac{z_n}{\sqrt{\sigma_n}}\widetilde{v}_n(\xi)\right\rangle_{\mathcal{Y}} d\xi \quad \forall (z_n) \in Z_b.$$

A simple estimate shows $|\langle \iota f, (z_n)\rangle| \leqslant \|\iota f\|_{L^1(\mathbb{R}_+;\mathcal{Y})}\|(z_n)\|_{Z_b}$ and hence, ιf is a functional on Z_b. The estimate $\|\iota f\|_{Z_b'} \leqslant \|f\|_{\widetilde{Z}_b}$ moreover shows the continuity of the embedding ι. To conclude the injectivity of ι, observe that $\langle \iota f, (z_n)\rangle = 0$ for all $(z_n) \in Z_b$ is equivalent to $\int_0^\infty \langle \widetilde{f}(\xi), w(\xi)\rangle_{\mathcal{Y}} d\xi = 0$ for all $w \in W^{1,1}(\mathbb{R}_+;\mathcal{Y}) \cap \operatorname{ran}\mathfrak{H}$, which implies $\widetilde{f} \in (\operatorname{ran}\mathfrak{H})^\perp$. Hence, f is the zero functional on $\operatorname{dom} A_b^*$ if $\iota f = 0$.

In order to prove the first part of *b)*, it suffices now to show that $A_b|_{\ell_2} e_j \in \widetilde{Z}_b$ for all $j \in \mathbb{N}$. Choose an arbitrary (z_n) in the set $\widetilde{Z} = \Sigma^{1/2}\operatorname{dom} A_o^*$, which was shown to be a core for A_b^* in the proof of Theorem 5.5 *c)*. Remembering the formula (7.3b) we choose $y \in W_0^{1,2}(\mathbb{R}_+;\mathcal{Y})$ with $\pi_{\overline{\operatorname{ran}\mathfrak{H}}} y = \widetilde{V}\Sigma^{-1/2}(z_n)$

and have

$$\begin{aligned}\langle A_b|_{\ell_2}e_j,(z_n)\rangle_{(\operatorname{dom}A_b^*)',\operatorname{dom}A_b^*} &= \langle e_j, A_b^*(z_n)\rangle_{\ell_2} = \langle e_j, \Sigma^{1/2}A_o^*\Sigma^{-1/2}(z_n)\rangle_{\ell_2}\\ &= \langle e_j, -\Sigma^{1/2}\widetilde{V}^*\tfrac{d}{d\xi}y\rangle_{\ell_2}\\ &= \left\langle \widetilde{V}\Sigma^{1/2}e_j, -\tfrac{d}{d\xi}y\right\rangle_{L^2(\mathbb{R}_+;\mathcal{Y})}\\ &= \int_0^\infty \left\langle \sqrt{\sigma_j}\tfrac{d}{d\xi}\widetilde{v}_j(\xi), y(\xi)\right\rangle_{\mathcal{Y}} d\xi.\end{aligned}$$

Here, we have used partial integration between a $W^{1,1}$- and a $W_0^{1,2}$-function, which can be justified by approximation with smooth functions. The equation above shows that $A_b|_{\ell_2}e_j$ is an element of $\widetilde{Z}_b$ with $\widetilde{A_b|_{\ell_2}e_j} = \sqrt{\sigma_j}\frac{d}{d\xi}\widetilde{v}_j$. Now it is merely a matter of definition to plug e_i into $\langle \iota A_b|_{\ell_2}e_j, e_i\rangle$ and get (4.4a).

To show the first equality in (4.4b), we observe that the equation (8.7) and (7.4) imply that $B_b u \in \widetilde{Z}_b$ with $\widetilde{B_b u} = hu$ for all u. Hence the embedding ι gives

$$\begin{aligned}\langle B_b u, e_i\rangle_{Z_b',Z_b} &= \frac{1}{\sqrt{\sigma_i}}\int_0^\infty \langle h(\xi)u, \widetilde{v}_i(\xi)\rangle_{\mathcal{Y}} d\xi = \frac{1}{\sqrt{\sigma_i}}\langle u, (H^*\widetilde{v}_i)(0)\rangle_{\mathcal{U}}\\ &= \frac{1}{\sqrt{\sigma_i}}\langle u, \widetilde{u}_i(0)\rangle_{\mathcal{U}} = b_i(u),\end{aligned}$$

where the representation of H^* given in [8, Lemma 5.3.3] was used. The second equality in (4.4b) uses the Lebesgue extension of B_b', which is defined as

$$(B_b')_L := \lim_{t\to 0}\frac{1}{t}\int_0^t (\mathfrak{B}_b^* x)(-\tau)d\tau.$$

Since $\operatorname{dom}(B_b')_L$ is by definition the set where this limit exists, we conclude from

$$\mathfrak{B}_b^* e_j = \mathfrak{B}^* SV\Sigma^{-1/2}e_j = \frac{1}{\sqrt{\sigma_j}}\mathfrak{B}^*\mathfrak{C}^*\widetilde{v}_i = \sqrt{\sigma_j}\widetilde{u}_j \quad \in W^{1,1}(\mathbb{R}_-;\mathcal{U})$$

that $e_i \in \operatorname{dom}(B_b')_L$ for all $i \in \mathbb{N}$ and

$$\langle u, (B_b')_L e_j\rangle_{\mathcal{U}} = \left\langle u, \sqrt{\sigma_j}\widetilde{u}_j(0)\right\rangle_U = b_j(u)(0)$$

Finally, by (7.5), the set $\Sigma^{-1/2}Z_b$ is contained in the domain of C_o since $W^{1,1}$-functions are continuous and therefore have a Lebesgue point at zero. Hence $Z_b \subset \operatorname{dom} C_b = \Sigma^{1/2}\operatorname{dom} C_o$ and again with (7.5) we conclude

$$C_b e_j = C_o\Sigma^{1/2}e_j = \lim_{t\to 0}\frac{1}{t}\int_0^t \frac{e_j}{\sqrt{\sigma_j}}\widetilde{v}_j(\tau)d\tau = \frac{1}{\sqrt{\sigma_j}}\widetilde{v}_j(0) = c_i.$$

□

Proof of Theorem 5.7. The proof is completely analogous to the previous one. We only give the necessary definitions

$$Z_o := \left\{(z_n)\in\ell_2 \,\middle|\, \widetilde{V}(z_n)\in W^{1,1}(\mathbb{R}_+;\mathcal{Y})\right\}, \quad \|(z_n)\|_{Z_o} := \|\widetilde{V}(z_n)\|_{W^{1,1}(\mathbb{R}_+;\mathcal{Y})},$$

and

$$\widetilde{Z}_o := \left\{ f \in (\operatorname{dom} A_o^*)' \;\middle|\; \begin{array}{l} \exists \widetilde{f} \in L^1(\mathbb{R}_+;\mathcal{Y}) \quad \forall (x_n) \in \operatorname{dom} A_o^* : \\ \langle f, (x_n)\rangle = \displaystyle\int_0^\infty \Big\langle \widetilde{f}(\xi), y(\xi)\Big\rangle_{\mathcal{Y}} d\xi \;\text{ for some} \\ y \in W_0^{1,2}(\mathbb{R}_+;\mathcal{Y}) \text{ with } \pi_{\overline{\operatorname{ran}\mathfrak{H}}} y = \widetilde{V}(x_n). \end{array} \right\}.$$

□

Remark 9.1. In the case when $h \in L^2(\mathbb{R}_+;\mathcal{B}(\mathcal{U},\mathcal{Y})$ and the Schmidt vectors therefore satisfy $\widetilde{v}_i \in W^{1,2}(\mathbb{R}_+;\mathcal{Y})$ and $\widetilde{u}_i \in W^{1,2}(\mathbb{R}_+;\mathcal{U})$, we have to make the following adaptions: In the proof of Theorem 5.6, respectively 5.7, replace $W^{1,1}$ by $W^{1,2}$ and L^1 by L^2 in the definitions of Z_b and $\widetilde{Z}_b$, respectively Z_o and $\widetilde{Z}_o$. This way, Z_o becomes $\operatorname{dom} A_o$ and $\widetilde{Z}_o$ becomes ℓ_2. In Theorem 5.6, Z_b becomes $\Sigma^{1/2}\operatorname{dom} A_o$ and $\widetilde{Z}_b$ is $\Sigma^{-1/2}\ell_2 := \{(y_n) \,|\, \Sigma^{1/2}(y_n) \in \ell_2\}$, with $\widetilde{(y_n)} = \widetilde{V}\Sigma^{1/2}(y_n)$ being the required L^2-function for each $(y_n) \in \Sigma^{-1/2}\ell_2$. The unitarity of $\widetilde{V}$ than implies

$$\langle \iota(y_n), x_n)\rangle_{Z_b',Z_b} = \langle \Sigma^{1/2}(y_n), \Sigma^{-1/2}(x_n)\rangle_{\ell_2}.$$

10. Proof of Theorem 5.8

Proof of Theorem 5.8. We proceed similar to the proof of Theorem 5.6. Define

$$Z := \left\{ z \in \operatorname{ran}\mathfrak{C}^* \;\middle|\; \widetilde{\mathfrak{C}}^{-*} z \in \operatorname{ran}\mathfrak{H} \cap W^{1,1}(\mathbb{R}_+;\mathcal{Y}) \right\}$$

with norm $\|z\|_Z := \|\widetilde{\mathfrak{C}}^{-*} z\|_{W^{1,1}(\mathbb{R}_+;\mathcal{Y})}$. The space

$$\widetilde{Z} := \left\{ f \in (\operatorname{dom}\widetilde{A}^*)' \;\middle|\; \begin{array}{l} \exists \widetilde{f} \in L^1(\mathbb{R}_+;\mathcal{Y}) \quad \forall y \in W_0^{1,2}(\mathbb{R}_+;\mathcal{Y}) \subset \operatorname{dom}\widetilde{A}^* : \\ \langle f, \widetilde{\mathfrak{C}}^* y\rangle = \displaystyle\int_0^\infty \Big\langle \widetilde{f}(\xi), y(\xi)\Big\rangle_{\mathcal{Y}} d\xi \end{array} \right\}$$

with norm $\|f\|_{\widetilde{Z}} := \|\widetilde{f}\|_{L^1(\mathbb{R}_+;\mathcal{Y})}$ is continuously embedded into Z' via the injection

$$\iota : \widetilde{Z} \to Z', \qquad \langle \iota f, z\rangle := \int_0^\infty \Big\langle \widetilde{f}(\xi), \widetilde{\mathfrak{C}}^{-*} z(\xi)\Big\rangle_{\mathcal{Y}} d\xi \quad \forall z \in Z.$$

Furthermore, for all $\forall y \in W_0^{1,2}(\mathbb{R}_+;\mathcal{Y})$ we have $A^*\mathfrak{C}^* y = \mathfrak{C}^*\dot{y} \in \operatorname{ran}\mathfrak{C}^*$ and

$$\begin{aligned} \langle A|_X R u_i, \mathfrak{C}^* y\rangle_{(\operatorname{dom} A^*)',\operatorname{dom} A^*} &= \langle A|_X \mathfrak{B}\widetilde{u}_i, \mathfrak{C}^* y\rangle_{(\operatorname{dom} A^*)',\operatorname{dom} A^*} \\ &= \langle \mathfrak{C}^{-1}\mathfrak{C}\mathfrak{B}\widetilde{u}_i, A^*\mathfrak{C}^* y\rangle_X \\ &= \langle \mathfrak{C}\mathfrak{B}\widetilde{u}_i, \mathfrak{C}^{-*}A^*\mathfrak{C}^* y\rangle_{L^2(\mathbb{R}_+;\mathcal{Y})} \\ &= -\langle \widetilde{v}_i, \pi_{(\ker\mathfrak{C}^*)^\perp}\tfrac{d}{d\xi} y\rangle_{L^2(\mathbb{R}_+;\mathcal{Y})} \\ &= \langle \tfrac{d}{d\xi}\widetilde{v}_i, y\rangle_{L^1(\mathbb{R}_+;\mathcal{Y}),L^\infty(\mathbb{R}_+;\mathcal{Y})}. \end{aligned}$$

This shows $A|_X Ru_i \in \widetilde{Z} \subset Z'$. Together with $\mathfrak{C}^{-*}Sv_j = \widetilde{v}_j \in \operatorname{ran}\mathfrak{H} \cap W^{1,1}(\mathbb{R}_+;\mathcal{Y})$, this shows *b)*. Our definition of the embedding ι leads to

$$\begin{aligned}\langle A|_X Ru_i, Sv_j\rangle_{\widetilde{Z},Z} &= \langle A|_X \mathfrak{B}\widetilde{u}_i, \mathfrak{C}^*\widetilde{v}_j\rangle_{\widetilde{Z},Z} = \langle \iota A|_X \mathfrak{B}\widetilde{u}_i, \mathfrak{C}^*\widetilde{v}_j\rangle_{Z',Z}\\ &= \langle \tfrac{d}{d\xi}\widetilde{v}_i, \mathfrak{C}^{-*}\mathfrak{C}^*\widetilde{v}_j\rangle_{L^1(\mathbb{R}_+;\mathcal{Y}),L^\infty(\mathbb{R}_+;\mathcal{Y})} = \frac{\sqrt{\sigma_i}}{\sqrt{\sigma_j}}a_{ij}.\end{aligned}$$

With the adjoint $B' \in \mathcal{B}(\operatorname{dom} A^*;\mathcal{U})$ of B we obtain for all $u \in \mathcal{U}$ and $y \in W_0^{1,2}(\mathbb{R}_+;\mathcal{Y})$

$$\begin{aligned}\langle Bu, \mathfrak{C}^*y\rangle_{(\operatorname{dom} A^*)',\operatorname{dom} A^*} &= \langle u, B'\mathfrak{C}^*y\rangle_{\mathcal{U}} = \langle u, (H^*y)(0)\rangle_{\mathcal{U}}\\ &= \left\langle u, \int_0^\infty h^*(\tau)y\,d\tau\right\rangle_{\mathcal{U}} = \int_0^\infty \langle h(\tau)u, y\rangle_{\mathcal{Y}}d\tau.\end{aligned}$$

This shows that $\operatorname{ran} B \subset \widetilde{Z} \subset Z'$ and therefore

$$\begin{aligned}\langle Bu, Sv_i\rangle_{\widetilde{Z}_b,Z_b} &= \langle \iota Bu, \mathfrak{C}^*\widetilde{v}_i\rangle_{Z_b',Z_b} = \langle hu, \mathfrak{C}^{-*}\mathfrak{C}^*\widetilde{v}_i\rangle_{L^1(\mathbb{R}_+;\mathcal{Y}),L^\infty(\mathbb{R}_+;\mathcal{Y})}\\ &= \sqrt{\sigma_i}b_i(u).\end{aligned}$$

For the alternative representation, we observe that $\mathfrak{B}^*\mathfrak{C}^*\widetilde{v}_i = \sigma_i\widetilde{u}_i$, which is in $W^{1,1}(\mathbb{R}_-;\mathcal{U})$. This implies $\mathfrak{C}^*\widetilde{v}_i \in \operatorname{dom}(B')_L$ for all $i \in \mathbb{N}$ and

$$b_i u = \sqrt{\sigma_i}\langle u, \widetilde{u}_i(0)\rangle_{\mathcal{U}} = \frac{1}{\sqrt{\sigma_i}}\langle u, (H^*\widetilde{v}_j)(0)\rangle_{\mathcal{U}} = \frac{1}{\sqrt{\sigma_j}}\langle u, (B')_L\mathfrak{C}^*\widetilde{v}_j\rangle_U.$$

Finally, the equality

$$\mathfrak{C}Ru_j = \mathfrak{C}\mathfrak{B}\widetilde{u}_j\widetilde{v}_j \in W^{1,1}(\mathbb{R}_+;\mathcal{Y})$$

shows that Ru_j is in the domain of C and

$$CRu_j = \lim_{t\to 0}\frac{1}{t}\int_0^t(\mathfrak{C}Ru_j)(\tau)d\tau = \lim_{t\to 0}\frac{1}{t}\int_0^t \widetilde{v}_j(\tau)d\tau = \widetilde{v}_j(0) = \sqrt{\sigma_i}c_i.$$

□

11. Pseudo-similarity in the minimal case

Proposition 11.1. *With the preliminaries of Theorem* 5.2 *and, additionally, minimality of the system* $(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D})$*, the following holds true: The operator* $\mathcal{T} = V^*S^*$ *is a pseudo-similarity transformation defined on* X*. The inverse pseudo-similarity transformation* $S^{-*}V : \operatorname{ran} V^*S^* \subset \ell_2 \to \ell_2$ *is exactly the closure* $\overline{RU\Sigma^{-1}}$ *of the operator* $\mathcal{T}^+ : \Sigma\ell_2 \subset \ell_2 \to X$*. Accordingly, the domain of* $\overline{RU\Sigma^{-1}}$ *equals* $\operatorname{ran} V^*S^*$ *in this case.*

Proof. The properties of a pseudo-similarity required in Definition 2.5 are readily verified using that

$$\mathfrak{A}_o(t)(x_n) = V^*S^*\mathfrak{A}S^{-*}V(x_n) \qquad \forall (x_n) \in \operatorname{ran} V^*S^*,$$

which follows by a density argument, since both operator coincide on $\Sigma\ell_2 \subset \operatorname{ran} V^*S^*$. Only the claim about the inverse transformation needs to be

proven at full length: In the minimal case, the singular value decomposition (5.3) reads

$$S^*R|_{(\ker R)^\perp} = V\Sigma U^*, \tag{11.1}$$
$$U \in \mathcal{B}\left(\ell_2; (\ker R)^\perp\right), \quad V \in \mathcal{B}\left(\ell_2; (\ker S)^\perp\right),$$

because the density of $\operatorname{ran} R$ implies that $\overline{\operatorname{ran} S^*R} = \overline{\operatorname{ran} S^*} = (\ker S)^\perp$, and $\ker S^*R = \ker R$ due to the injectivity of S^*. Another consequence of this is that V^*S^* is injective and the inverse $S^{-*}V : \operatorname{ran} V^*S^* \subset \ell_2 \to X$ of this bounded operator is a closed operator. Because $\pi_{(\ker S^*)^\perp}$ is obsolete, it follows from (7.1) that $S^{-*}V$ must be a closed extension of $RU\Sigma^{-1}$ to $\operatorname{ran} V^*S^*$. In order to prove that it is the smallest closed extension, i.e. the adjoint of the adjoint of $RU\Sigma^{-1}$, we have to determine the domains of these operators.

Our first claim is that $\operatorname{dom}(RU\Sigma^{-1})^* = \operatorname{ran} S$. So pick an arbitrary $y \in \operatorname{ran} S$. Then for all $(x_n) \in \Sigma\ell_2$ holds

$$\begin{aligned}\langle RU\Sigma^{-1}(x_n), y\rangle_X &= \langle RU\Sigma^{-1}(x_n), SS^{-1}y\rangle_X = \langle S^*RU\Sigma^{-1}(x_n), S^{-1}y\rangle_{X_S}\\ &= \langle V\Sigma U^*U\Sigma^{-1}(x_n), S^{-1}y\rangle_{X_S} = \langle V(x_n), S^{-1}y\rangle_{X_S}\\ &\leqslant \|S^{-1}y\|_{X_S}\|(x_n)\|_{\ell_2},\end{aligned}$$

and hence $y \in \operatorname{dom}(RU\Sigma^{-1})^*$. For the other inclusion, take an arbitrary $y \in \operatorname{dom}(RU\Sigma^{-1})^*$. Then there exists some constant K with

$$K\|(x_n)\| \geqslant \langle RU\Sigma^{-1}(x_n), y\rangle_X = \langle \Sigma^{-1}(x_n), U^*R^*y\rangle_{\ell_2} \qquad \forall (x_n) \in \Sigma\ell_2,$$

showing that U^*R^*y lies in the domain of the self-adjoint operator Σ^{-1}, which is $\Sigma\ell_2$. (In particular, it follows $(RU\Sigma^{-1})^* = \Sigma^{-1}U^*R^*$.) As a consequence of the controllability assumption, R^* is injective, so y can be written as $y = R^{-*}U\Sigma x$ for some $x \in \ell_2$. Adjoining equation (11.1) gives

$$R^*S|_{(\ker S)^\perp} = (S^*R|_{(\ker R)^\perp})^* = U\Sigma V^*.$$

and consequently $y \in R^{-*} \operatorname{ran} R^*S|_{(\ker S)^\perp} = \operatorname{ran} S$. So we have indeed shown $\operatorname{ran} S = \operatorname{dom}(RU\Sigma^{-1})^*$.

Since we have already seen that $S^{-*}V$ with its domain $\operatorname{ran} V^*S^*$ is an extension of $RU\Sigma^{-1}$, it suffices to show

$$\begin{aligned}\operatorname{ran} V^*S^* &\subset \operatorname{dom} \overline{RU\Sigma^{-1}}\\ &= \left\{(x_n) \in \ell_2 \,|\, \exists K > 0\ \forall y \in \operatorname{ran} S : \langle (x_n), (RU\Sigma^{-1})^*y\rangle_{\ell_2} \leqslant K\|y\|_X\right\}\end{aligned}$$

in order to complete the proof. So take an arbitrary $(x_n) \in \operatorname{ran} V^*S^*$. Then for all $y \in \operatorname{ran} S$ holds $U^*R^*y \in \Sigma\ell_2$ and, with the representation $\Sigma^{-1} =$

$V^*S^{-1}R^{-*}U|_{\Sigma\ell_2}$, we obtain

$$\begin{aligned}\langle (x_n), (RU\Sigma^{-1})^*y\rangle_{\ell_2} &= \langle V^*S^*\, S^{-*}V(x_n), (RU\Sigma^{-1})^*y\rangle_{\ell_2}\\ &= \langle S^{-*}V(x_n), SV\Sigma^{-1}U^*R^*y\rangle_X\\ &= \langle S^{-*}V(x_n), SS^{-1}R^{-*}R^*y\rangle_X\\ &= \langle S^{-*}V(x_n), y\rangle_X\\ &\leqslant \|S^{-*}V(x_n)\|_X\|y\|_X.\end{aligned}$$

This is what we needed to show. □

Proposition 11.2. *If in Theorem* 5.4 *the original system is minimal, then* T^+ *is closable and its closure*

$$\overline{T^+} : \{x \in \ell_2 : \Sigma^{1/2}x \in \operatorname{ran} V^*S^*\} \subset \ell_2 \to X, \qquad T^+ = RU\Sigma^{-1/2}$$

is a pseudo-similarity transformation. Its inverse transformation is the closure of T*, i.e.,*

$$\overline{T} : \{x \in X : V^*S^*x \in \Sigma^{1/2}\ell_2\} \subset X \to \ell_2, \qquad T := \Sigma^{-1/2}V^*S^*.$$

In the proof we make use of the following result which is straightforward to show:

Lemma 11.3. *Let* X, Y, Z *be Hilbert spaces and* $G \in \mathcal{B}(X;Y)$*. Furthermore let* $F : \operatorname{dom} A \subset Y \to Z$ *be a densely defined, closable operator and* $\overline{F} := F^{**}$ *its closure. If we define the operators* $FG : \{x \in X : Gx \in \operatorname{dom} F\} \subset X \to Z$ *and* $\overline{F}G : \{x \in X : Gx \in \operatorname{dom} \overline{F}\} \subset X \to Z$*, then the closure of* FG *equals* $\overline{F}G$*.*

Proof of Proposition 11.2. For T^+ to be closable it is necessary and sufficient that $\operatorname{dom} T^{+*} \subset \ell_2$ be dense. As a consequence of the observability assumption, $\ker S^* = \{0\}$, the range of S is dense, and we will show that the latter is contained in the domain of T^{+*}, which by definition is

$$\operatorname{dom} T^{+*} = \{y \in X |\ \exists k > 0\ \forall (x_n) \in \operatorname{dom} T^+ :\ \langle T^+(x_n), y\rangle_X \leqslant k\|(x_n)\|_{\ell_2}\}.$$

So let $y \in \operatorname{ran} S$ be arbitrary. Then $U^*R^*y \in \Sigma\ell_2$ and for $(x_n) \in \operatorname{dom} T^+ = \Sigma^{1/2}\ell_2$ holds

$$\begin{aligned}|\langle T^+(x_n), y\rangle_X| &= |\langle \Sigma^{-1/2}(x_n), U^*R^*y\rangle_{\ell_2}| = |\langle (x_n), \Sigma^{-1/2}U^*R^*y\rangle_{\ell_2}|\\ &\leqslant \|\Sigma^{-1/2}U^*R^*y\|_{\ell_2}\|(x_n)\|_{\ell_2},\end{aligned}$$

showing $y \in \operatorname{dom} T^{+*}$. So we may take the closure $\overline{T^+}$ of the operator $T^+ = RU\Sigma^{-1/2}$ which, according to Lemma 11.3, is equal to the concatenation

$$\overline{RU\Sigma^{-1}}\Sigma^{1/2} : \{(x_n) \in \ell_2 : \Sigma^{1/2}(x_n) \in \operatorname{ran} V^*S^*\} \subset \ell_2 \to X$$

of the two injective operators $\overline{RU\Sigma^{-1}} : \operatorname{ran} V^*S^* \subset \ell_2 \to X$ and $\Sigma^{1/2} \in \mathcal{B}(\ell_2)$. Thus, $\overline{T^+}$ is injective as well. We remark that its range is in general smaller then the range of $\overline{RU\Sigma^{-1}}$, which is X. More precisely, we have $\operatorname{ran} \overline{T^+} = \{x \in X : V^*S^*x \in \Sigma^{1/2}\ell_2\}$. For if $x \in \operatorname{ran} \overline{T}$, then $x = \overline{RU\Sigma^{-1}}\Sigma^{1/2}(y_n)$ for some $(y_n) \in \operatorname{dom} \overline{T^+}$ and, remembering that V^*S^* is the inverse of $\overline{RU\Sigma^{-1}}$,

$VS^*x = \Sigma^{1/2}(y_n)$. On the other hand, if x is such that $(y_n) = V^*S^*x \in \Sigma^{1/2}\ell_2$, then $(y_n) \in \operatorname{ran} V^*S^* = \operatorname{dom} \overline{RU\Sigma^{-1}}$ and $x = \overline{RU\Sigma^{-1}}(y_n)$ is in the range of $\overline{T^+} = \overline{RU\Sigma^{-1}}\Sigma^{1/2}$. And with this we have shown that the inverse of $\overline{T^+}$ is $\Sigma^{1/2}V^*S^*$.

The invariance properties of Definition 2.5 are not quite as easy to see as in Proposition 11.1. We need to show the $\mathfrak{A}$-invariance of $\operatorname{dom}\overline{T}$. That is, for all $x \in X$ with the property that $V^*S^*x \in \Sigma^{1/2}\ell_2$ must hold $V^*S^*\mathfrak{A}(t)x \in \Sigma^{1/2}\ell_2$. Let $x \in \operatorname{dom}\overline{T}$, then $(y_n) := V^*S^*x \in \Sigma\ell_2$, and hence

$$V^*S^*\mathfrak{A}(t)x = V^*S^*\mathfrak{A}(t)S^{-*}V(y_n) = \mathfrak{A}_o(t)(y_n),$$

which is again an element of $\Sigma^{1/2}\ell_2$, because by Lemma 8.1, $\mathfrak{A}_o$ leaves this space invariant. For the $\mathfrak{A}_b$ invariance of $\operatorname{ran}\overline{T}$ observe that $(y_n) := \Sigma^{1/2}(x_n) \in \operatorname{ran} V^*S^*$ for all $(x_n) \in \operatorname{ran}\overline{T}$ and hence

$$\mathfrak{A}_b(t)(x_n) = \Sigma^{-1/2}V^*S^*\mathfrak{A}(t)RU\Sigma^{-1}\Sigma^{1/2}(x_n) = \Sigma^{-1/2}V^*S^*\mathfrak{A}(t)S^{-*}V(y_n).$$

The right hand side shows that this is again an element of $\operatorname{ran}\overline{T}$. It is obvious that $\mathfrak{B}$ maps into $\operatorname{dom}\overline{T}$ since it maps into $\Sigma\ell_2$, and that $\Sigma^{-1/2}V^*S^*\mathfrak{B}$ maps into $\operatorname{dom}\overline{T^+}$. □

References

[1] A.C. Antoulas, *Approximation of large-scale dynamical systems*, Advances in design and control, Society for Industrial and Applied Mathematics, Philadelphia/Pennsylvania, 2005.

[2] R.F. Curtain, *On model reduction for control design for distributed parameter systems*, in Research Directions in Distributed Parameter Systems, R. Smith and M. Demetriou, eds., SIAM, Boston, MA, 2003, pp. 95–121.

[3] R.F. Curtain and K. Glover, *Balanced realisations for infinite-dimensional systems*, Oper. Theory Adv. Appl., 19 (1986), pp. 87–104.

[4] R.F. Curtain and A.J. Sasane, *Compactness and nuclearity of the Hankel operator and internal stability of infinite-dimensional state linear systems*, Int. J. Control, 71 (2001), pp. 1260–1270.

[5] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics Volume 194, Springer, New York, 2000.

[6] K. Glover, R.F. Curtain, and J.R. Partington, *Realization and approximation of linear infinite-dimensional systems with error bounds*, SIAM J. Control Optim., 26 (1988), pp. 863–898.

[7] K. Glover, J. Lam, and J.R. Partington, *Rational approximation of a class of infinite-dimensional systems i: Singular values of Hankel operators*, Math. Control Signals Systems, 3 (1990), pp. 325–344.

[8] C. Guiver, *Model Reduction by Balanced Truncation*, PhD thesis, University of Bath, Department of Mathematical Sciences, 2012.

[9] C. Guiver and M. Opmeer, *Model reduction by balanced truncation for systems with nuclear hankel operators*, SIAM J. Control Optim., (2013). to appear.

[10] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, Heidelberg and New York, 1980.

[11] B.C. Moore, *Principal component analysis in linear systems: controllability, observability, and model reduction*, IEEE Trans. Automat. Control, 26 (1981), pp. 17–32.

[12] R.J. Ober, *Infinite dimensional balanced realizations and their approximation*, in Proc. 25th IEEE Conf. Decision Control, vol. 25, Athens, 1986, IEEE, pp. 155–156.

[13] ———, *A parametrization approach to infinite-dimensional balanced systems and their approximation*, IMA J. Math Control Inform., 4 (1987), pp. 263–279.

[14] ———, *Balanced parametrization of classes of linear systems*, SIAM J. Control Optim., 29 (1991), pp. 1251–1287.

[15] R.J. Ober and S. Montgomery-Smith, *Bilinear transformation of infinite dimensional state space systems and balanced realizations of nonrational transfer functions*, SIAM J. Control Optim., 28 (1990), pp. 438–465.

[16] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, New York, London, 1972.

[17] T. Reis, M.R. Opmeer, and W. Wollner, *Finite-rank ADI iteration for operator Lyapunov equations*, SIAM J. Control Optim., (2013). to appear.

[18] O. Staffans, *Well-posed Linear systems*, vol. 103 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2005.

[19] M.S. Tombs and I. Postlethwaite, *Truncated balanced realization of a stable non-minimal state-space system*, Int. J. Control, 46 (1987), pp. 1319–1330.

[20] G. Weiss, *Representation of shift invariant operators on L_2 by $\mathcal{H}_\infty$ transfer functions, an elementary proof, a generalization to L_p and a counterexample for L_∞*, Math. Control Signals Systems, 4 (1991), pp. 193–203.

[21] G. Weiss, *Transfer functions of regular linear systems. I. characterizations of regularity*, Trans. Amer. Math. Soc., 342 (1994), pp. 827–854.

[22] G. Weiss and M. Tusznak, *Observation and Control for Operator Semigroups*, Birkhäuser Advanced Texts Basler Lehrbücher, Birkhäuser, Basel, 2009.

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