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FUNNEL CONTROL FOR THE BOUNDARY CONTROLLED HEAT EQUATION

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Abstract. We consider an output regulation problem for a single input single output system with dynamics described by the heat equation on some bounded domain $\Omega \subset \mathbb{R}^d$ with sufficiently smooth boundary. The input is formed by Neumann boundary control, the output is the surface integral of the state at the boundary. We show that the funnel controller can be applied to this system in order to track a given output reference signal within a prespecified performance funnel.

Key words. infinite-dimensional linear systems, adaptive control, high-gain output feedback, funnel control, heat equation, boundary control

AMS subject classifications. 93C20, 93C40, 35K05

1. Introduction. Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ be a bounded domain with uniformly C^2 boundary $\partial\Omega$ [1, Chap. 4]. Consider the following heat equation with Neumann boundary control and a Dirichlet-like boundary observation:

$$\begin{aligned} \frac{\partial x}{\partial t}(\xi,t) &= \Delta_\xi x(\xi,t), && (\xi,t) \in \Omega \times \mathbb{R}_{>0}, \\ u(t) &= \partial_\nu x(\xi,t), && (\xi,t) \in \partial\Omega \times \mathbb{R}_{>0}, \\ y(t) &= \int_{\partial\Omega} x(\xi,t)\,\mathrm{d}\sigma_\xi, && (\xi,t) \in \partial\Omega \times \mathbb{R}_{>0}, \\ x(\xi,0) &= x_0(\xi), && \xi \in \Omega. \end{aligned} \tag{1.1}$$

Our aim is to apply output feedback control in order to achieve that the output signal $y : \mathbb{R}_{>0} \to \mathbb{R}$ tracks a given reference signal $y_{\mathrm{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ in a way that for a given function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, the error

$$e(t) = y(t) - y_{\mathrm{ref}}(t) \tag{1.2}$$

evolves inside the *performance funnel*

$$\mathcal{F}_\varphi := \{(t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \mid \varphi(t)\,|e| < 1\}\ . \tag{1.3}$$

see Figure 1.1. Specifically, the transient behavior is supposed to satisfy

$$\|e(t)\| < 1/\varphi(t) \quad \forall\, t > 0.$$

In particular, if φ is chosen so that $\varphi(t) \geq 1/\lambda$ for all t sufficiently large, then the error remains smaller than $1/\lambda$ for these t.

To ensure the above control objective, we introduce the *funnel controller*:

$$\boxed{u(t) = -\frac{\varphi(t)^2}{1-\varphi(t)^2 e(t)^2}\, e(t).} \tag{1.4}$$

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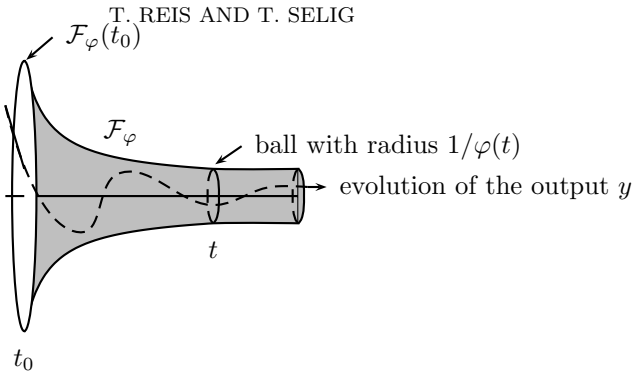

FIG. 1.1. *Error evolution in the funnel* $\mathcal{F}_\varphi$

Intuitively, in order to maintain the output evolution within the funnel, the control signal $u(t)$ in (1.4) reaches high values if the error $e(t)$ is close to the funnel boundary $\pm\varphi(t)^{-1}$, driving it back towards zero. On the other hand, if the output signal is close enough to the reference signal, the gain is also small.

This control law has shown to be feasible for linear time-invariant input-state-output systems governed by ordinary differential equations (ODEs), i.e.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t), \end{aligned}$$

with the following properties:

- input and output dimensions are equal, i.e. $B, C^\top \in \mathbb{R}^{n\times m}$ for some $n, m \in \mathbb{N}$;
- the system has *strict relative degree one*, i.e. CB is invertible;
- the zero dynamics of the system are asymptotically stable, i.e. all trajectories $x(\cdot)$ and $u(\cdot)$ of the system that result in a trivial output $y \equiv 0$ tend to zero.

For this class the funnel controller not only achieves that the output evolves in the funnel; the state trajectory is also bounded [15–17]. Feasibility of the funnel controller has moreover been shown for linear differential-algebraic systems [3–5] and nonlinear ODE systems [14, 18]. These approaches have in common that the feasibility was proven on the basis of canonical forms under the group action of state space transformation. In particular, the *Byrnes-Isidori form* [18, 20] can be gainfully used to show that the funnel controller is feasible for ODE systems with asymptotically stable zero dynamics and relative degree one. The transformation to Byrnes-Isidori form has recently been considered for a class of infinite-dimensional linear systems of type (3.1) in [19]. The (possibly unbounded) operator $A : D(A) \subset X \to X$ was assumed to generate a strongly continuous semigroup on the state space X and, in the case of relative degree one, the operators B and C were assumed to map into $D(A^*)$ and from $D(A)$, respectively. These additional boundedness properties have been essential for the existence of the Byrnes-Isidori form.

The boundary controlled heat equation (1.1) can be formulated as an infinite-dimensional linear system. However, due to the fact that control and observation are at the boundary, the operators B and C are now so-called *unbounded control and observation operators*. That is, B maps to the space $D(A^*)' \supset X$, and C is defined on a proper subspace of X [7]. Consequently, no transformation to Byrnes-Isidori form is possible. The product CB whose invertibility indicates the relative degree one property cannot even be formed!

In this article we show that funnel control is nevertheless possible for the heat

equation system (1.1). We show that, under certain assumptions on smoothness and boundedness on the funnel function φ and the reference signal y_{ref}, the funnel controller accomplishes the objective. We will moreover show that the funnel control signal $u : \mathbb{R}_{>0} \to \mathbb{R}$ as in (1.4) is a bounded and continuous function. Our proof is based on modal approximation of the input-output map by finite-dimensional linear systems with asymptotically stable zero dynamics and relative degree one. We will show that funnel control is feasible for these truncated systems and that the sequence of solutions to the closed loop truncated systems contains a convergent subsequence. The limit of this subsequence will solve a nonlinear Volterra equation that represents the input-output behavior of the heat equation system (1.1) under the funnel feedback (1.4). This solution results in a well-defined input signal $u \in L^2_{\mathrm{loc}}(\mathbb{R}_{>0})$. Inserting this signal into the heat equation (1.1) yields a solution to the funnel controlled heat equation in the sense of well-posed linear systems. We will then show that this solution x solves the partial differential equation formed by (1.1), (1.2), (1.4) in a stronger sense and that it has additional regularity and boundedness properties, namely $x(\cdot, t) \in W^{1,2}(\Omega)$ for all $t > 0$ and $\sup_{t\geq 0} \|x(\cdot, t)\|_{L^2(\Omega)} < \infty$.

This article is organized as follows: In the remaining part of the introductory section, we collect the notation that is used throughout this work. We present our main result on feasibility of the funnel controller for the heat equation in Section 2. In Section 3 we collect some properties of the representation of the boundary controlled heat equation (1.1) as an infinite-dimensional linear system which are mostly taken from [7] and [22]. In particular, a representation of the input-output as a convolution mapping is crucial. It will be used in Section 4 to reformulate the funnel control problem as a nonlinear Volterra equation. We first analyze solvability of the Volterra equations corresponding to the funnel control problem for the modal truncated systems, and then discuss their limiting behavior. This section will contain the proof that the funnel control problem has a global solution in which the output evolves in the funnel, i.e. the essential part of the main result is proven in this section. It remains to prove that funnel control results in a bounded state trajectory and to discuss its regularity. This will be done in Section 5 which also contains the formal proof of our main theorem. We have two appendices: The first one contains crucial supplements on passivity of the truncated systems appearing in Section 4; the second one contains results on proportional output feedback that are needed to show regularity and boundedness in Section 5.2.

1.1. Nomenclature.

$\mathbb{N}$, $\mathbb{N}_0$		set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, resp.
$\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$	$=$	$[0,\infty)$, $(0,\infty)$, resp.
$\operatorname{Re}\lambda$, $\overline{\lambda}$		real part and complex conjugate of $\lambda \in \mathbb{C}$, resp.
$\mathbb{R}^{n\times m}$		the set of real $n \times m$ matrices
$M^\top$		transpose of the matrix (or vector) M
$\ker A$, $\operatorname{ran} A$		kernel and range of a linear operator A
$A\|_Y$		restriction of a mapping $A : X \to H$ to the subset $Y \subset X$
I		identity mapping
$\mathcal{B}(X,Y)$		the set of bounded linear operator from X to Y
$\rho(A)$, $\sigma(A)$		the resolvent set and spectrum of a linear operator A
$\ell^p(\mathbb{N})$, $\ell^p(\mathbb{N}_0)$		$p \in [1,\infty]$, the space of p-summable sequences $(a_k)_{k\in\mathbb{N}}$, $(a_k)_{k\in\mathbb{N}_0}$, resp.
$L^p(\Omega;X)$		$p \in [1,\infty]$, the Lebesgue space of measurable functions $x : \Omega \to X$, see [9, Chap. IV]
$L^p_{\mathrm{loc}}(\Omega;X)$		space of measurable functions from Ω to X that are locally in L^p
$L^p(\Omega)$, $L^p_{\mathrm{loc}}(\Omega)$	$=$	$L^p(\Omega;\mathbb{C})$, $L^p_{\mathrm{loc}}(\Omega;\mathbb{C})$, resp.
$\mathcal{C}(\Omega;X)$		the set of continuous functions from Ω to X
$\mathcal{C}(\Omega)$	$=$	$\mathcal{C}(\Omega;\mathbb{C})$
$\mathcal{BUC}(\Omega)$	$=$	$\{\ f \in L^\infty(\Omega)\ \mid f$ is uniformly continuous $\}$
$W^{k,p}(\Omega)$		$p \in [1,\infty]$, $k \in \mathbb{N}$ (or $p \in [1,\infty)$, $k \in \mathbb{R}_{\geq 0}$), (fractional) Sobolev space of functions $f : \Omega \to \mathbb{C}$, see [1, Chap. 3] or [13]

The scalar product $\langle\cdot,\cdot\rangle$ in a Hilbert space H is defined to be linear in the first and antilinear in the second component. On the dual space H' we define multiplication such that $(\lambda y)(x) := \overline{\lambda} y(x)$ for $y \in H'$ and $x \in H$. With this definition the dual pairing $\langle y, x\rangle := \overline{y(x)}$ for $y \in H'$ and $x \in H$ becomes linear in the first and anti-linear in the second component.

In this article $\Omega \subset \mathbb{R}^d$ is always a bounded open set with a uniformly $\mathcal{C}^2$-boundary $\partial\Omega$ [1, Chap. 4]. Integration on the surface of this manifold is indicated by σ_ξ. For $\xi \in \partial\Omega$ we denote by $\nu(\xi)$ the outward normal of $\partial\Omega$ and by $\partial_\nu x(\xi)$ the directional derivative of some function $x \in L^2(\mathbb{R}^d)$ along ν at the point ξ, whenever it is well-defined. By ∇x, Δx we denote the (distributional) gradient, respectively Laplacian of x.

For the notion of (strongly continuous, contractive, analytic, bounded, exponentially stable) semigroup we refer to [24]. A definition and properties of sesquilinear forms can be found in [21].

2. The main result. Our goal is to steer system (1.1) via the control u in such a way that the output signal y is close to a desired reference signal y_{ref}. For this signal

we always assume

$$y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0}).$$

That is, the reference signal is Lipschitz continuous.

Note that for initial values $x(0) \in L^2(\Omega)$ the output $y(0)$ is not defined, so neither is the control law (1.2), (1.4). In order to apply funnel control we therefore adjust the class of funnels in such a way that the control u is zero on some small initial interval. To ensure this we assume that the function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defining the performance funnel via (1.3) satisfies for some $\gamma_0 > 0$

$$\varphi \in \Phi_{\gamma_0} := \left\{ \varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0}) \;\middle|\; \begin{array}{l} \varphi|_{[0,\gamma_0)} \equiv 0\,, \text{ and} \\ \forall\, \delta > 0 \;:\; \inf\{\; \varphi(t) \;\mid t > \gamma_0 + \delta \;\} > 0 \end{array} \right\}.$$

In other words,

$$\varphi \in \Phi := \bigcup_{\gamma_0 > 0} \Phi_{\gamma_0}.$$

Remark 2.1.

(i) The existence of $\gamma_0 > 0$ such that φ vanishes on $[0, \gamma_0)$ means that the funnel control $u(\cdot)$ as in (1.4) is inactive for a (short) while after zero. This additional assumption does not have to be made in funnel control for ODE or DAE systems [5, 15].

The practical interpretation is that the system has to settle down first: The funnel controller makes use of the fact that, after a (short) while $\gamma_0 > 0$, the spatial temperature distribution becomes smooth (cf. Lemma 3.3).

(ii) Define $\lambda := \inf_{t \geq \gamma_0 + \delta} \varphi(t)$. Then the error $e(t)$ is forced to be smaller than $1/\lambda$ for all $t > \gamma_0 + \delta$ because the reciprocal of φ describes the funnel boundary.

(iii) Each $\phi \in \Phi_{\gamma_0}$ is globally Lipschitz continuous since $\Phi_{\gamma_0} \subset W^{1,\infty}(\mathbb{R}_{\geq 0})$. Furthermore, for all $\delta > 0$ the expression $\inf\{\; \varphi(t) \;\mid t > \gamma + \delta \;\}$ is positive, whence the function $\varphi|_{[\gamma+\delta,\infty)}(\cdot)^{-1}$ is globally Lipschitz continuous.

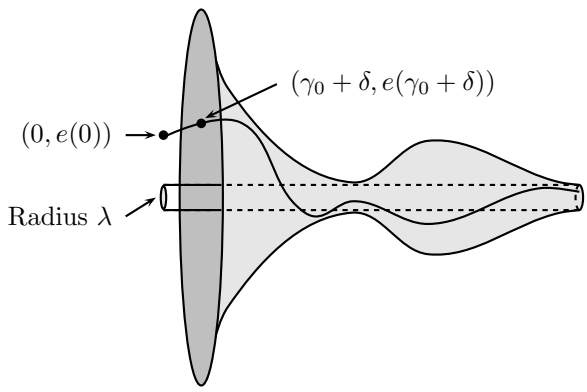

Fig. 2.1. *Error evolution in the funnel $\mathcal{F}_\varphi$ with "width ∞" in $[0, \gamma_0]$, i.e. $\varphi|_{[0,\gamma_0]} \equiv 0$*

The funnel controller (1.4) applied to (1.1) yields for any reference signal a partial differential equation with nonlinear boundary conditions. Our main result states that this evolution equation has a unique global, bounded solution.

THEOREM 2.2. *Given a reference signal* $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0})$ *and initial data* $x_0 \in L^2(\Omega;\mathbb{R})$*, choose any* $\varphi \in \Phi$ *and denote the performance funnel associated to* φ *via* (1.3) *by* $\mathcal{F}_\varphi$*. Then there exists a function* $x : \Omega \times [0,\infty) \to \mathbb{C}$ *with the following properties:*

(i) $x(\cdot,t) \in W^{1,2}(\Omega)$ *for all* $t > 0$*, and* $\sup_{t\geq 0} \|x(\cdot,t)\|_{L^2(\Omega)} < \infty$*, and*

(ii) the function $y : \mathbb{R}_{>0} \to \mathbb{R}$ *with*

$$y(t) = \int_{\partial\Omega} x(\xi,t)d\sigma_\xi \qquad \forall t \in \mathbb{R}_{>0}$$

is continuous on $\mathbb{R}_{>0}$ *and bounded on any interval* $[\delta,\infty)$ *with* $\delta > 0$*.*

(iii) The tracking error $e := y - y_{\mathrm{ref}}$ *evolves within the funnel* $\mathcal{F}_\varphi$ *and is uniformly bounded away from the funnel boundary, meaning*

$$\exists\, \varepsilon > 0\ \forall\, t > 0\ :\ \varphi(t)^2 e(t)^2 \leq 1 - \varepsilon\,. \tag{2.1}$$

(iv) The control function $u : \mathbb{R}_{>0} \to \mathbb{R}$ *with*

$$u(t) := \frac{\varphi(t)^2}{1 - \varphi(t)^2\, e(t)^2} \cdot e(t) \qquad \forall t \in \mathbb{R}_{>0}$$

is bounded and uniformly continuous, i.e. $u \in \mathcal{BUC}(\mathbb{R}_{>0})$*.*

(v) For all $\psi \in W^{1,2}(\Omega)$ *and* $t > 0$*, the scalar function* $\langle x(\cdot,t),\psi(\cdot)\rangle_{L^2(\Omega)}$ *is differentiable with*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\langle x(\cdot,t),\psi\rangle_{L^2(\Omega)} = -\langle \nabla x(\cdot,t),\nabla\psi\rangle_{L^2(\Omega)} + u(t)\cdot\int_{\partial\Omega}\overline{\psi(\xi)}d\sigma_\xi \qquad \forall t \in \mathbb{R}_{\geq 0}.$$

REMARK 2.3. *Let us elucidate the points of this theorem one by one.*

(i) This means that the state trajectory is bounded in the $L^2(\Omega)$ *norm. In fact we, can even say more: The* $W^{1,2}(\Omega)$ *norm of* x *is bounded on any interval* $[\delta,\infty)$ *with* $\delta > 0$*, and if* x_0 *is in* $W^{1,2}(\Omega)$*, then this norm is bounded on* $\mathbb{R}_{\geq 0}$*, see Proposition 5.3.*

(ii) For general $x_0 \in L^2(\Omega;\mathbb{R})$ *the output signal* y *cannot be defined at the point zero. That is why the function* y *cannot be bounded on* $\mathbb{R}_{\geq 0}$ *in general. However, if* x_0 *is in* $W^{1,2}(\Omega;\mathbb{R})$*, then part (i) of this remark and the fact that the mapping in (ii) is continuous from* $W^{1,2}(\Omega;\mathbb{R})$ *to* $\mathbb{C}$ *imply that* y *is bounded on* $\mathbb{R}_{\geq 0}$*.*

(iii) Note that (2.1) *implies the existence of some* $\varepsilon' > 0$ *such that* $|e(t)|^2 \leq \varphi(t)^{-2} - \varepsilon'$ *for all* $t \geq \gamma_0$*. Equivalently, there exists some* $\varepsilon'' > 0$ *such that* $|e(t)| \leq \varphi(t)^{-1} - \varepsilon''$ *for all* $t \geq \gamma_0$*. Since* $\varphi(\cdot)^{-1}$ *describes the funnel boundary, this shows that the error evolves within the funnel and has a positive distance to the funnel boundary. In this sense our tracking goal is achieved. Note that, by formally setting* $\frac{1}{\infty} = 0$*, we see that these inequalities also hold true on the whole positive real axis.*

(iv) Note that the uniform bound in (iii) guarantees that the control u *is well-defined and evolves in the bounded interval* $\varepsilon^{-1}\|\varphi\|_\infty \cdot [-1,1]$*.*

(v) This means that x, y *and* u *solve the partial differential equation* (1.1) *in a weak sense. This weak formulation is obtained by multiplying* (1.1) *with a test function* ψ *and using Gauss' Theorem. In this weak formulation, the second summand on the right represents the boundary control. In fact, a stronger statement than (iii) holds: The function* $t \mapsto (\psi \mapsto \langle x(\cdot,t),\psi\rangle_{L^2(\Omega)})$ *is differentiable with respect to the topology of* $W^{1,2}(\Omega)'$*.*

Parts (ii)-(iv) of this theorem will be shown in Section 5.1, and the proof of (i) follows in Section 5.2. The formal proof of the whole Theorem 2.2 in Section 5.3 collects these results together and shows that partial differential equation in part (v) is fulfilled.

3. The heat equation as infinite-dimensional linear system. In [7] the partial differential equation (1.1) was shown to fit into the framework of infinite-dimensional regular well-posed linear systems. Further investigation of this system has been carried out in [22]. We briefly recap the results from [7, 22] that are crucial for the present article.

By taking $x(t) := x(\cdot, t) \in L^2(\Omega)$, the heat equation (1.1) can be interpreted as an infinite-dimensional linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{3.1a}$$

$$y(t) = Cx(t) \tag{3.1b}$$

on the state space $X := L^2(\Omega)$ with A, B and C defined in the following:
a) $A : D(A) \subset X \to X$ with

$$Ax = \Delta x \qquad \forall\, x \in D(A) = \left\{\, x \in W^{2,2}(\Omega) \;\middle|\; \partial_\nu x|_{\partial\Omega} = 0 \,\right\}; \tag{3.2a}$$

b) $B \in \mathcal{B}(\mathbb{C}, W^{\theta,2}(\Omega)')$ for all $\theta \in (\frac{1}{2}, 1]$ is defined by

$$\langle Bu, \varphi\rangle = u \cdot \int_{\partial\Omega} \overline{\varphi(\xi)} \mathrm{d}\sigma_\xi \qquad \forall\, \varphi \in W^{\theta,2}(\Omega),\ u \in \mathbb{C}; \tag{3.2b}$$

c) $C : D(C) \to \mathbb{C}$ with $D(C) \supset W^{\theta,2}(\Omega)$ for all $\theta \in (\frac{1}{2}, 1]$ is defined by

$$Cx = \int_{\partial\Omega} x(\xi) \mathrm{d}\sigma_\xi \qquad \forall\, x \in D(C). \tag{3.2c}$$

Note that B and C are well-defined by the fact that for $\theta \in (\frac{1}{2}, 1]$ there exists a continuous linear trace operator mapping $W^{\theta,2}(\Omega)$ into $L^2(\partial\Omega)$ [13, Thm. 4.24 (i)]. The precise domain $D(C)$ is defined in [7, Eq. (6.9)]. For our purposes it suffices to know that C is well-defined on $W^{\theta,2}(\Omega)$ for all $\theta \in (\frac{1}{2}, 1]$. For these values of θ the operator $B : \mathbb{C} \to W^{\theta,2}(\Omega)'$ is the adjoint operator of $C|_{W^{\theta,2}(\Omega)}$ in the sense that

$$\langle Bu, \varphi\rangle = \langle u, C\varphi\rangle \qquad \forall u \in \mathbb{C},\ \varphi \in W^{\theta,2}(\Omega).$$

Lemma 3.1 ([22, Lem. 2.4 and Lem. 2.2 (ii)]). *Let A be defined as in* (3.2a). *The resolvent of A is compact and there is a real valued sequence $(\lambda_k)_{k\in\mathbb{N}_0}$ such that*
a) (λ_k) is nondecreasing, $\lambda_0 = 0$, $\lambda_1 > 0$, and $\lim_{k\to\infty} \lambda_k = \infty$;
b) $\sigma(A) = \{-\lambda_k \mid k \in \mathbb{N}_0\}$.
Further, there is an orthonormal basis $(v_k)_{k\in\mathbb{N}_0}$ of $L^2(\Omega)$ with $v_k \in D(A)$ for all $k \in \mathbb{N}_0$, and

$$Ax = -\sum_{k=0}^{\infty} \lambda_k \langle x, v_k\rangle_{L^2(\Omega)} \cdot v_k \qquad \forall x \in D(A). \tag{3.3}$$

The operator A generates a contractive, analytic semigroup $\mathfrak{A} : \mathbb{R}_{\geq 0} \to \mathcal{B}\left(L^2(\Omega)\right)$, which can be extended to $\mathcal{B}\left(D(A)'\right)$.

If we use the extension of $\mathfrak{A}$ to $\mathcal{B}(D(A)')$, then for $x_0 \in L^2(\Omega)$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ the variation of constants formula

$$x(t) := \mathfrak{A}(t)x_0 + \int_0^t \mathfrak{A}(t-\tau)Bu(\tau)\mathrm{d}\tau \qquad \forall\, t \in \mathbb{R}_{\geq 0} \tag{3.4}$$

is well defined as B maps into $W^{1,2}(\Omega)' \subset D(A)'$. The function $x(\cdot) : \mathbb{R}_{\geq 0} \to D(A)'$ defined by (3.4) is called *mild solution* of (3.1a). The following result shows that the mild solution (3.4) is even pointwise in X and moreover, $x(t) \in D(C)$ for almost all $t \in \mathbb{R}_{\geq 0}$.

THEOREM 3.2 ([7, Cor. 1]). *Let $X = L^2(\Omega)$ and the operators A, B and C as in* (3.2) *be given. Then the following holds true:*

(i) For all $u \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$, $x_0 \in X$, the function defined in (3.4) *fulfills*
   a) $x(t) \in X$ for all $t \in \mathbb{R}_{\geq 0}$;
   b) $x(t) \in D(C)$ for almost all $t \in \mathbb{R}_{\geq 0}$.

(ii) For all $t \in \mathbb{R}_{\geq 0}$, there exists some $c_t \in \mathbb{R}_{\geq 0}$, such that for all $u \in L^2([0,t])$, $x_0 \in X$, the solutions of (3.1) *fulfill*

$$\|y(\cdot)\|_{L^2([0,t])} + \|x(t)\|_X \leq c_t \cdot \left(\|u(\cdot)\|_{L^2([0,t])} + \|x_0\|_X\right).$$

The above statement means that the system (3.1) is *well-posed*. This basically comprises four properties, namely the boundedness of the semigroup $\mathfrak{A}(\cdot)$ on each compact interval $[0,t]$ (which is guaranteed anyway by its strong continuity), as well as the boundedness of the input-to-state map $\mathfrak{B}_t : L^2([0,t]) \to X$, the state-to-output map $\mathfrak{C}_t : X \to L^2([0,t])$, and the input-output map $\mathfrak{D}_t : L^2([0,t]) \to L^2([0,t])$, which are defined by

$$\begin{aligned} \mathfrak{B}_t u &= \int_0^t \mathfrak{A}(t-\tau)Bu(\tau)\mathrm{d}\tau, & \mathfrak{C}_t x &= (t' \mapsto C\mathfrak{A}(t')x), \\ \mathfrak{D}_t u &= \left(t' \mapsto C\int_0^{t'} \mathfrak{A}(t'-\tau)Bu(\tau)\mathrm{d}\tau\right). \end{aligned} \tag{3.5}$$

The latter two operators naturally extend to the infinite-time state-to-output and input-to-output mappings

$$\begin{aligned} \mathfrak{C}:\quad & X \to L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0}), & \mathfrak{D}:\quad & L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0}) \to L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0}), \\ & x \mapsto (t \mapsto C\mathfrak{A}(t)x), & & u \mapsto \left(t \mapsto C\int_0^t \mathfrak{A}(t-\tau)Bu(\tau)\mathrm{d}\tau\right). \end{aligned} \tag{3.6}$$

For any input function $u \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ and initial value $x_0 \in X$, the state $x \in L^2(\Omega)$ and output $y \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ of the system (3.1) are defined by

$$\begin{aligned} x(t) &:= \mathfrak{A}(t)x_0 + \mathfrak{B}_t u|_{[0,t]} \qquad \forall t \in \mathbb{R}_{\geq 0}, \\ y &:= \mathfrak{C}x_0 + \mathfrak{D}u. \end{aligned} \tag{3.7}$$

LEMMA 3.3. *Let A and C be defined as in* (3.2). *Then the following holds true:*

(i) For all $k \in \mathbb{N}$, $\delta \in \mathbb{R}_{>0}$ and $x \in L^2(\Omega)$, the semigroup $\mathfrak{A}(\cdot)$ generated by A fulfills $\mathfrak{A}(t)x \in D(A^k)$.

(ii) For all $\delta \in \mathbb{R}_{>0}$ and $x \in L^2(\Omega)$, the infinite-time state-to-output map fulfills

$$\mathfrak{C}x|_{[\delta,\infty)} \in W^{1,\infty}([\delta,\infty)).$$

Proof. By Lemma 3.1 the semigroup $\mathfrak{A}$ is bounded and analytical. Therefore (i) holds, see e.g. [10, Chap. II, Thm. 4.6]. In order to see (ii), let $\delta > 0$ and $x \in L^2(\Omega)$. Then by (i), there holds $\mathfrak{A}(\delta)x \in D(A)$. Since, by [10, Chap. II, Thm. 5.2], the restriction of $\mathfrak{A}(\cdot)$ to $D(A)$ is a bounded semigroup on $D(A)$, there holds

$$\mathfrak{A}(\cdot)|_{[\delta,\infty)}x = \mathfrak{A}(\cdot - \delta)|_{[\delta,\infty)}\mathfrak{A}(\delta)x \in L^\infty([\delta,\infty); D(A)).$$

Furthermore, by $\mathfrak{A}(\delta) \in D(A^2)$, we obtain by the same argumentation that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{A}(\cdot)|_{[\delta,\infty)}x = A\mathfrak{A}(\cdot)|_{[\delta,\infty)}x \in L^\infty([\delta,\infty); D(A)).$$

Since C as in (3.2) fulfills $C \in \mathcal{B}(D(A), \mathbb{R})$, we have

$$\begin{aligned} \mathfrak{C}x|_{[\delta,\infty)} &= C\mathfrak{A}(\cdot)x|_{[\delta,\infty)} \in L^\infty([\delta,\infty)), \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{C}x|_{[\delta,\infty)} &= CA\mathfrak{A}(\cdot)x|_{[\delta,\infty)} \in L^\infty([\delta,\infty)). \end{aligned}$$

□

System (3.2) possesses so-called *transfer function*, cf. [29, 31] and the bibliographies therein.

Definition 3.4. *Let the triple (A, B, C) consist the operators in* (3.2)*. Let $r(A, B, C) \subset \sigma(A)$ be the set of removable singularities of the function*

$$\rho(A) \to \mathbb{C}, \qquad s \mapsto C(sI - A)^{-1}B,$$

and let $D(G) = \rho(A) \cup r(A, B, C)$. We define the transfer function $G : D(G) \to \mathbb{C}$ *of (A, B, C) to be the analytic extension of $C(sI - A)^{-1}B$.*

Remark 3.5. *Existence and uniqueness of the analytic extension of G to $D(G)$ is guaranteed by Riemann's theorem [23, Thm. 10.21] and* $\operatorname{ran}(sI - A)^{-1}B \subset D(C)$ *for all $s \in \rho(A)$, see [22, Sec. 3].* The transfer function of our system is regular in the sense of [27].

Proposition 3.6. *[7] The transfer function of the operators (A, B, C) given by* (3.2) *is* regular *with zero feedthrough, which means*

$$\lim_{s>0, s\to\infty} G(s) = 0.$$

We collect some properties of the transfer function $C(sI - A)^{-1}B$ of (3.2). It admits a partial fraction expansion, which will be the basis for further investigations.

Lemma 3.7 ([22, Thm. 3.6]). *Let A, B and C be defined as in* (3.2) *and let (λ_k), (v_k) be as in Lemma 3.1. Define*

$$c_k := \left| \int_{\partial\Omega} v_k(\xi)\, \mathrm{d}\sigma_\xi \right|^2 \quad \forall k \in \mathbb{N}_0 \qquad \text{and} \qquad J_c := \{k \in \mathbb{N}_0 \mid c_k \neq 0\}. \tag{3.8}$$

Then for all $s \in \rho(A)$ the transfer function of (A, B, C) fulfills

$$G(s) = \sum_{k=0}^{\infty} \frac{c_k}{s + \lambda_k} = \sum_{k \in J_c} \frac{c_k}{s + \lambda_k}. \tag{3.9}$$

Furthermore, we have $0 \in J_c$, and

$$\left(\frac{c_k}{\lambda_k}\right) \in \ell_1(\mathbb{N}). \tag{3.10}$$

This partial fraction expansion translates into a useful representation of the input output map $\mathfrak{D}$ in the time domain.

LEMMA 3.8. *Let A, B and C be defined as in* (3.2). *Then, with sequences $(c_k)_{k\in\mathbb{N}_0}$, $(\lambda_k)_{k\in\mathbb{N}_0}$ as in Lemma 3.7, the input-output map $\mathfrak{D}_t \in \mathcal{B}(L^2([0,t]))$ defined in* (3.5) *fulfills*

$$(\mathfrak{D}_t u)(t') = \int_0^{t'} \sum_{k=0}^{\infty} c_k \mathrm{e}^{\lambda_k \tau} u(t'-\tau)\mathrm{d}\tau \qquad \forall u \in L^2([0,t]),\ \ t' \in [0,t]. \tag{3.11}$$

Proof. Let $u \in L^2([0,t])$ be given. Then, by extending u by zero on (t,∞), we can regard $u(\cdot)$ as an element of $L^2(\mathbb{R}_{\geq 0})$. In particular, we have $\mathrm{e}^{-\cdot}u(\cdot) \in L^2(\mathbb{R}_{\geq 0})$. Now define $y(\cdot) = \mathfrak{D}u \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$. Then, by using Theorem 3.2 (iii), we obtain that $\mathrm{e}^{-\cdot}y(\cdot) \in L^2(\mathbb{R}_{\geq 0})$. Define the set $\mathbb{C}_{+,\alpha} = \{\ s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha\ \}$. Using Lemma 3.7 we obtain that the Laplace transforms of $u(\cdot)$ and $y(\cdot)$ are related by

$$\widehat{y}(s) = \sum_{k=0}^{\infty} \frac{c_k \cdot \widehat{u}(s)}{s+\lambda_k} \qquad \forall s \in \mathbb{C}_{+,1}.$$

An application of the inverse Laplace transform now leads to

$$y = \mathcal{L}^{-1}\left(s \mapsto \sum_{k=0}^{\infty} \frac{c_k \cdot \widehat{u}(\cdot)}{s+\lambda_k} \right).$$

Since $\sup_{s\in\mathbb{C}_{+,0}} \left| \frac{c_k}{s+\lambda_k} \right| = \frac{c_k}{\lambda_k}$ for all $k \in \mathbb{N}$, the series $s \mapsto \sum_{k=0}^{\infty} \frac{c_k}{s+\lambda_k}$ converges absolutely in all the Hardy spaces

$$\mathcal{H}_\infty(\mathbb{C}_{+,\alpha}) := \{\ f : \mathbb{C}_{+,\alpha} \to \mathbb{C} \mid f \text{ is bounded and holomorphic }\},\qquad \alpha \in \mathbb{R}_{>0},$$

which are provided with the supremum norm, see [8, Sec. A.6.3]. Hence the order of inverse Laplace transform and summation may be interchanged. By further using

$$\mathcal{L}^{-1}\left(s \mapsto \frac{c_k \cdot \widehat{u}(s)}{s+\lambda_k} \right) = \left(t' \mapsto c_k \int_0^{t'} \mathrm{e}^{-\lambda_k(t'-\tau)} u(\tau)\mathrm{d}\tau \right) \qquad \forall k \in \mathbb{N}_0,$$

we obtain

$$y = \left(t' \mapsto \sum_{k=0}^{\infty} c_k \int_0^{t'} \mathrm{e}^{-\lambda_k(t'-\tau)} u(\tau)\mathrm{d}\tau \right). \tag{3.12}$$

Since, by Young's inequality [6, Thm. 3.9.4], there holds

$$\left\| \int_0^{\cdot} \mathrm{e}^{-\lambda_k(\cdot-\tau)} u(\tau)\mathrm{d}\tau \right\|_{L^2([0,t])} \leq \frac{\|u(\tau)\|_{L^2([0,t])}}{\lambda_k} \qquad \forall k \in \mathbb{N},$$

the series in (3.12) converges absolutely in $L^2([0,t])$, and consequently, the order of integration and summation may be interchanged. This proves (3.11). ∎

This convolution is the basis for our results. In fact, throughout the next Section we will only assume that we are given a convolution kernel with the same properties as in Lemmas 3.7 and 3.8, see Assumption 1.

4. The funnel control problem as a nonlinear Volterra equation. In this section we consider an inhomogeneous Volterra equation which is motivated by the heat equation. However, the results are independent of Section 3 and based solely on the following assumption, which we make throughout Section 4.

ASSUMPTION 1. *Let the sequences* $(c_k)_{k\in\mathbb{N}_0}$ *and* $(\lambda_k)_{k\in\mathbb{N}_0}$ *satisfy*
a) $c_k \geq 0$ *and* $\lambda_k \geq 0$ *for all* $k \in \mathbb{N}_0$*;*
b) $c_0 > 0$*;*
c) $(\lambda_k)_{k\in\mathbb{N}_0}$ *is monotonically increasing with* $\lambda_0 = 0$, $\lambda_1 > 0$ *and and* $\lim_{k\to\infty} \lambda_k = \infty$*;*
d)

$$\left(\frac{c_k}{\lambda_k}\right) \in \ell_1(\mathbb{N}). \tag{4.1}$$

LEMMA 4.1. *Under Assumption 1 the series*

$$h = \left(t \mapsto \sum_{k=0}^{\infty} c_k \mathrm{e}^{\lambda_k t}\right) \tag{4.2}$$

fulfills $h \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ *with*

$$\|h\|_{L^1([0,t])} = c_0 t + \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k}(1 - \mathrm{e}^{-\lambda_k t}).$$

Moreover, the operator $\mathfrak{D} : L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ *with*

$$\mathfrak{D}u = \left(t' \mapsto \int_0^{t'} \sum_{k=0}^{\infty} h(t'-\tau)u(\tau)\mathrm{d}\tau\right) \tag{4.3}$$

is well-defined. For each $t \geq 0$*, the restriction* $\mathfrak{D}_t := \mathfrak{D}|_{[0,t]}$ *fulfills* $\operatorname{ran}\mathfrak{D}_t \subset \mathcal{BUC}([0,t])$ *and*

$$\|\mathfrak{D}_t\|_{\mathcal{B}(L^\infty([0,t]))} = \|h\|_{L^1([0,t])} = c_0 t + \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k}(1 - \mathrm{e}^{-\lambda_k t}). \tag{4.4}$$

Proof. Using nonnegativity of c_k, a simple calculation gives

$$\int_0^t |c_k \mathrm{e}^{-\lambda_k \tau}|\mathrm{d}\tau = \int_0^t c_k \mathrm{e}^{-\lambda_k \tau}\mathrm{d}\tau = \frac{c_k}{\lambda_k}(1 - \mathrm{e}^{-\lambda_k t}) \qquad \forall k \in \mathbb{N}$$

and

$$\int_0^t |c_0 \mathrm{e}^{-\lambda_0 \tau}|\mathrm{d}\tau = \int_0^t c_0 \mathrm{e}^{-\lambda_0 \tau}\mathrm{d}\tau = c_0 t.$$

Hence, by (4.1), the series in (4.2) converges in $L^1([0,t])$, and we may interchange the order of integration and summation to obtain

$$\|h\|_{L^1([0,t])} = \int_0^t \sum_{k=0}^{\infty} c_k \mathrm{e}^{-\lambda_k \tau} = c_0 t + \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k}(1 - \mathrm{e}^{-\lambda_k t}).$$

The assertion $\mathfrak{D}_t \in \mathcal{B}(L^\infty([0,t]))$ with

$$\|\mathfrak{D}_t\|_{\mathcal{B}(L^\infty([0,t]))} \leq \|h\|_{L^1([0,t])}$$

then follows from Young's inequality [6, Thm. 3.9.4]. On the other hand, since for $u \equiv 1$ holds

$$(\mathfrak{D}_t u)(t) = \int_0^t \sum_{k=0}^\infty c_k \mathrm{e}^{\lambda_k \tau} \mathrm{d}\tau = \|h\|_{L^1([0,t])},$$

we obtain (4.4). The statement $\operatorname{ran} \mathfrak{D}_t \subset \mathcal{BUC}([0,t])$ follows from [6, Cor. 3.9.6] and the fact that $[0,t]$ is compact. ◻

We are going to analyze a Volterra type equation that is motivated by our original funnel control problem for the heat equation in the following way: For $u \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ and $x_0 \in L^2(\Omega)$, the output of the partial differential equation model (1.1) is given by

$$y = (\mathfrak{D}u) + (\mathfrak{C}x),$$

with operators $\mathfrak{D}$, $\mathfrak{C}$ as in (3.6). By Lemma 3.7 and Lemma 3.8, the input-output map $\mathfrak{D}$ is of the form (4.3) and satisfies the assumptions of this section. Now subtracting the reference signal on both sides and defining the error $e := y - y_{\mathrm{ref}}$, and regarding

$$f := (\mathfrak{C}x) - y_{\mathrm{ref}} \tag{4.5}$$

as an inhomogeneity, the funnel feedback defined by (1.2) and (1.4) gives rise to

$$e = (\mathfrak{D}(k \cdot e)) + f, \qquad \text{with} \quad k(t) = -\frac{\varphi(t)^2}{1 - \varphi(t)^2 e(t)^2}.$$

This is a nonlinear, inhomogeneous Volterra equation for which we now try to find a solution e.

THEOREM 4.2. *Under Assumption 1, let $t_0 > 0$ and $f \in W^{1,\infty}([t_0,\infty))$. Choose $\varphi \in \Phi$ such that $\varphi(t_0) > 0$ and $|f(t_0)| < \frac{1}{\varphi(t_0)}$. Then the equation*

$$e(t) = \int_{t_0}^t h(t-\tau) \cdot k(\tau, e(\tau)) e(\tau) \mathrm{d}\tau + f(t), \qquad t \geq t_0 \tag{4.6a}$$

with

$$k(t,e) = -\frac{\varphi(t)^2}{1 - \varphi(t)^2 \cdot e^2} \tag{4.6b}$$

has a bounded, global solution $e \in \mathcal{BUC}([t_0,\infty))$, which is uniformly bounded away from the funnel boundary in the sense that

$$\exists\, \varepsilon' > 0\, \forall t \geq t_0 : \ |e(t)|^2 \leq \varphi(t)^{-2} - \varepsilon'. \tag{4.7}$$

Before proving this result, we state a corollary. It contains the uniqueness of the solution and it states that one can start the funnel with an infinite radius, i.e. with $\varphi(\gamma_0) = 0$ at initial time $t_0 = \gamma_0$. In this case the assumption that the initial value of f lies within the funnel becomes redundant.

Corollary 4.3. *Under Assumption 1, let $\gamma_0 > 0$, $\varphi \in \Phi_{\gamma_0}$ and a function $f \in W^{1,\infty}([\gamma_0, \infty))$ be given. Then the equation*

$$e(t) = \int_{\gamma_0}^{t} h(t-\tau) \cdot k(\tau, e(\tau)) e(\tau) \mathrm{d}\tau + f(t), \qquad t \geq \gamma_0 \tag{4.8}$$

with k as in (4.6b) *has a unique global solution $e \in \mathcal{BUC}([\gamma_0, \infty))$. This solution is uniformly bounded away from the funnel boundary in the sense that*

$$\exists\, \varepsilon' > 0 \ \forall t > \gamma_0 \ : \ |e(t)|^2 \leq \varphi(t)^{-2} - \varepsilon' .$$

Proof. First of all it follows with standard fixed point arguments, see [12, Chap. 12, Thm. 1.1], that for sufficiently small $t_0 > \gamma_0$ there exists a unique solution $e_0 \in \mathcal{BUC}([\gamma_0, t_0])$ of (4.8). If t_0 is chosen small enough, the limit $\varphi(t) \to 0$ for $t \to \gamma_0$ guarantees that

$$\left| \int_{\gamma_0}^{t} h(t-\tau) \cdot k(\tau, e_0(\tau)) \cdot e_0(\tau) \mathrm{d}\tau + f(t) \right| < \frac{1}{\varphi(t)} \qquad \forall t \in (\gamma_0, t_0].$$

In particular, this implies that the function $\widetilde{f} \in W^{1,\infty}([t_0, \infty))$ defined by

$$\widetilde{f}(t) = \int_{\gamma_0}^{t_0} h(t-\tau) \cdot k(\tau, e_0(\tau)) \cdot e_0(\tau) \mathrm{d}\tau + f(t), \qquad t \geq t_0,$$

satisfies the prerequisites of Theorem 4.2. This gives rise to the existence of a solution $\widetilde{e} \in \mathcal{BUC}([t_0, \infty))$ of the Volterra integral equation

$$\widetilde{e}(t) = \int_{t_0}^{t} h(t-\tau) \cdot k(\tau, \widetilde{e}(\tau)) \cdot \widetilde{e}(\tau) \mathrm{d}\tau + \widetilde{f}(t), \qquad t \geq t_0$$

A simple calculation shows that the combined function

$$e(t) := \begin{cases} e_0(t), & t \in [\gamma_0, t_0), \\ \widetilde{e}(t), & t \in [t_0, \infty) \end{cases}$$

is bounded, uniformly continuous and solves (4.8).

In order to prove the uniqueness of e, we assume that for some $t \in [\gamma_0, \infty)$ there are $e_1, e_2 \in \mathcal{C}([0, t])$ that solve (4.8). This means in particular that

$$\varphi(t) e_1(t) < 1, \quad \varphi(t) e_2(t) < 1 \qquad \forall t \in [\gamma_0, t].$$

Define $t' := \inf \{ \ \tau \in [\gamma_0, t] \ | \ e_1(\tau) \neq e_2(\tau) \ \}$. We show that $t' < t$ leads to a contradiction. Pick $\varepsilon > 0$ such that for all τ in the compact interval $[\gamma_0, t]$, the following inequalities hold:

$$\varphi^2(\tau) e_1^2(\tau) \leq 1 - \varepsilon, \quad \varphi^2(\tau) e_2^2(\tau) \leq 1 - \varepsilon.$$

Further, choose δ such that

$$\int_0^{\delta} h(\tau) \, \mathrm{d}\tau < \frac{\varepsilon^4}{2 \|\varphi\|^2_{L^\infty([0,t])}}.$$

Then defining for $i \in \{1,2\}$ the abbreviations

$$u_i := (t \mapsto k(t, e_i(t)) \cdot e_i(t)) = -\frac{\varphi^2}{1-\varphi^2 e_i^2} \cdot e_i,$$

we obtain for all $t \in [t', t'+\delta]$

$$\begin{aligned}
&|e_1(t) - e_2(t)| \\
&\quad = \left| \int_{\gamma_0}^{t} h(t-\tau) u_1(\tau)\,\mathrm{d}\tau - \int_{\gamma_0}^{t} h(t-\tau) u_1(\tau)\,\mathrm{d}\tau \right| \\
&\quad \le \int_{t'}^{t} |h(t-\tau)||u_1(\tau) - u_2(\tau)|\,\mathrm{d}\tau \\
&\quad \le \int_{0}^{t-t'} |h(\tau)|\,\mathrm{d}\tau \cdot \sup_{\tau \in [t', t'+\delta]} |u_1(\tau) - u_2(\tau)| \\
&\quad \le \frac{\varepsilon^4}{2\|\varphi\|_\infty} \cdot \left\| \frac{\varphi^2 - \varphi^4 e_1 e_2}{(1-\varphi^2 e_1^2)(1-\varphi^2 e_2^2)} (e_1 - e_2) \right\|_{L^\infty([t', t'+\delta])} \\
&\quad \le \frac{\varepsilon^4}{2\|\varphi\|_\infty} \cdot \|\varphi\|_\infty^2 \underbrace{\|1 - \varphi^2 e_1 e_2\|_{L^\infty([t', t'+\delta])}}_{<1+1} \cdot \underbrace{\left\| \frac{1}{1-\varphi^2 e_1^2} \right\|_{L^\infty([t', t'+\delta])}}_{<\frac{1}{\varepsilon^2}} \\
&\qquad\qquad \cdot \underbrace{\left\| \frac{1}{1-\varphi^2 e_2^2} \right\|_{L^\infty([t', t'+\delta])}}_{<\frac{1}{\varepsilon^2}} \cdot \|e_1 - e_2\|_{L^\infty([t', t'+\delta])} \\
&\quad < \|e_1 - e_2\|_{L^\infty([t', t'+\delta])}.
\end{aligned}$$

Now taking the supremum of all $t \in [t', t'+\delta]$ leads to the contradiction

$$\|e_1 - e_2\|_{L^\infty([t', t'+\delta])} < \|e_1 - e_2\|_{L^\infty([t', t'+\delta])}.$$

□

The proof of Theorem 4.2 is divided into the following steps which will be carried out in Sections 4.1–4.3.

Step 1: Let $n \in \mathbb{N}$ and

$$h_n = \left(t \mapsto \sum_{k=0}^{n-1} c_k \mathrm{e}^{\lambda_k t} \right). \tag{4.9}$$

We show that there exists some bounded function $e^{\{n\}} \in \mathcal{C}([t_0, \infty))$ such that

$$e^{\{n\}}(t) = \int_{t_0}^{t} h_n(t-\tau) \cdot k(\tau, e^{\{n\}}(\tau)) e^{\{n\}}(\tau) \mathrm{d}\tau + f(t) \qquad \forall t \ge t_0. \tag{4.10}$$

We further show that all the functions $e^{\{n\}}$ have a positive distance to the funnel boundary which is independent of n, see (4.16).

Step 2: We show that the set

$$\left\{ e^{\{n\}} \;\middle|\; n \in \mathbb{N} \right\} \subset \mathcal{C}([t_0, \infty)) \cap L^\infty([t_0, \infty))$$

is equicontinuous.

Step 3: We show that the sequence $(e^{\{n\}})_{n \in \mathbb{N}}$ contains a uniformly convergent subsequence and that the limit of this sequence solves the nonlinear Volterra equation (4.6).

4.1. Step 1: Modal truncated systems. We show that the truncated equations (4.10) have solutions on $[t_0, \infty)$ with the property that $e^{\{n\}}$ evolves in the funnel, and the functions $|k^{\{n\}} e^{\{n\}}|$ are bounded from above by some constant independent of $n \in \mathbb{N}$. In the following Lemma we find a finite-dimensional state space realization of $\mathfrak{D}$. The special structure of the matrices in this system is often called Byrnes-Isidori form and facilitates the analysis of high gain feedback.

LEMMA 4.4. *Under Assumption 1, define h_n as in* (4.9) *and $\mathfrak{D}^{\{n\}}$ by*

$$\mathfrak{D}^{\{n\}} : L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}), \qquad \mathfrak{D}^{\{n\}} u = \left(t \mapsto \int_0^t h_n(t-\tau) u(\tau) \mathrm{d}\tau \right). \tag{4.11}$$

Then there exists some $\mathcal{A}_{12} \in \mathbb{R}^{1\times n}$ and a negative definite matrix $\mathcal{A}_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ such that, with the real numbers

$$\Gamma^{\{n\}} := \sum_{k=0}^{n-1} c_k, \qquad \mathcal{A}_{11} := -\sum_{k=0}^{n-1} c_k \lambda_k,$$

the matrix

$$\mathcal{A} := \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}$$

is negative semi-definite and the following is true: For $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ the equation $y = \mathfrak{D}^{\{n\}} u$ holds if and only if there is a function $z \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^n)$ that fulfills the ordinary differential equation

$$\begin{aligned} \dot{z}(t) &= \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} z(t) + \begin{bmatrix} \Gamma^{\{n\}} \\ 0_{n-1,1} \end{bmatrix} u(t), && z(0) = 0, \\ y(t) &= \begin{bmatrix} 1 & 0_{1,n-1} \end{bmatrix} z(t), && t \in \mathbb{R}_{\geq 0}. \end{aligned} \tag{4.12}$$

Proof. Define

$$A := \begin{bmatrix} -\lambda_0 & & \\ & \ddots & \\ & & -\lambda_{n-1} \end{bmatrix}, \qquad b := \begin{bmatrix} \sqrt{c_0} \\ \vdots, \\ \sqrt{c_{n-1}} \end{bmatrix}. \tag{4.13}$$

Then $h_n = (t \mapsto b^\top \mathrm{e}^{At} b)$, whence the operator $\mathfrak{D}^{\{n\}}$ fulfills

$$(\mathfrak{D}^{\{n\}} u)(t) = \int_0^t b^\top \mathrm{e}^{A(t-\tau)} b u(\tau) \mathrm{d}\tau \qquad \forall u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}), \quad t \in \mathbb{R}_{\geq 0}. \tag{4.14}$$

Now choose $\widetilde{U} := [\widetilde{u}_1, \ldots, \widetilde{u}_{n-1}]$ such that

$$U := \left[\frac{b}{\|b\|}, \widetilde{U} \right] \in \mathbb{R}^{n\times n}$$

is unitary. Then the inverse of $T := \frac{1}{\|b\|} U$ is given by

$$T^{-1} = \|b\| \left[\tfrac{b}{\|b\|}, \widetilde{U} \right]^\top.$$

A simple calculation shows that with

$$\mathcal{A}_{12} := \|b\|^{-1} b^\top A\widetilde{U}, \qquad \mathcal{A}_{22} := (\widetilde{U})^\top A\widetilde{U},$$

there holds

$$\begin{aligned}(\mathcal{A}, \mathcal{b}, \mathcal{c}) &:= \left(T^{-1}AT, T^{-1}b, b^\top T\right) \\ &= \left(\begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ (\mathcal{A}_{12})^\top & \mathcal{A}_{22} \end{bmatrix}, \begin{bmatrix} \Gamma^{\{n\}} \\ 0_{n-1,1} \end{bmatrix}, \begin{bmatrix} 1 \\ 0_{n-1,1} \end{bmatrix}^\top\right).\end{aligned} \tag{4.15}$$

It is clear from the definition of A that $\mathcal{A}_{22} = (\widetilde{U})^\top A\widetilde{U} \leq 0$. Suppose that

$$v^\top (\widetilde{U})^\top A\widetilde{U} v = 0 \quad \text{for some} \quad v \in \mathbb{R}^{n-1} \setminus \{0\}.$$

Then $\lambda_0 = 0$ yields $\widetilde{U}v \in \operatorname{span}\{e_1\}$ and since U is orthogonal, we conclude $v^\top (\widetilde{U})^\top b = 0$. Hence the first entry of b is zero, which contradicts the fact that $c_0 \neq 0$ by Lemma 3.7 a). Thus, $\mathcal{A}_{22}$ must be negative definite. The claim follows because the representation (4.14) of $\mathfrak{D}^{\{n\}}$ implies

$$(\mathfrak{D}^{\{n\}} u) = \left(t \mapsto \int_{t_0}^t b^\top T \mathrm{e}^{T^{-1}AT(t-\tau)} T^{-1} b u(\tau) \mathrm{d}\tau\right) \quad \forall u \in L^\infty_{\mathrm{loc}}([t_0, \infty)),\ t \in [t_0, \infty).$$

This is by (4.15) the variation of constants formula for the solution of the ODE (4.12). ∎

THEOREM 4.5. *Let $t_0 > 0$, $\varphi \in \Phi$ and $f \in W^{1,\infty}([t_0, \infty))$ satisfy $\varphi(t_0) > 0$ and $|f(t_0)| < \frac{1}{\varphi(t_0)}$. Let Assumption 1 hold and let h_n and k be defined by* (4.9) *and* (4.6b). *Then for all $n \in \mathbb{N}$, the equation*

$$e^{\{n\}}(t) = \int_{t_0}^t h_n(t-\tau) \cdot k(\tau, e^{\{n\}}(\tau)) e^{\{n\}}(\tau) \mathrm{d}\tau + f(t), \qquad t \geq t_0 \tag{4.10}$$

has a bounded, absolutely continuous solution $e^{\{n\}} : [t_0, \infty) \to \mathbb{R}$. There further exists a constant $\varepsilon' > 0$ independent of n such that

$$\forall n \in \mathbb{N}\ \forall t \geq t_0 :\ |e^{\{n\}}(t)|^2 \leq \varphi(t)^{-2} - \varepsilon'. \tag{4.16}$$

Proof. We define the auxiliary functions

$$f_0(t) := \begin{cases} \frac{t}{t_0} f(t_0), & t \in [0, t_0), \\ f(t), & t \geq t_0, \end{cases}$$

$$k_0(t, e) := \begin{cases} 0, & t \in [0, t_0), \\ -\frac{\varphi(t)^2}{1 - \varphi(t)^2 e^2}, & t \geq t_0 \end{cases}$$

and seek a solution to

$$e^{\{n\}}(t) = \left(\mathfrak{D} k_0(\,\cdot\,, e^{\{n\}}) \cdot e^{\{n\}}\right)(t) + f_0(t), \qquad t \in [0, \infty). \tag{4.17}$$

By Lemma 4.4 the equation (4.17) may equivalently be written as the initial value problem

$$\begin{aligned} \dot{z}(t) &= \mathcal{A} z(t) + \begin{bmatrix} \Gamma^{\{n\}} \\ 0_{n-1,1} \end{bmatrix} k_0(t, e^{\{n\}}(t)) e^{\{n\}}(t), \qquad z(0) = 0, \\ e^{\{n\}}(t) &= z_1(t) + f_0(t), \qquad t \geq 0 \end{aligned} \tag{4.18}$$

For $t \in [0, t_0]$ the functions $z(t) = 0_{n,1}$ and $e^{\{n\}}(t) = f_0(t)$ obviously solve this equation. For $t \geq t_0$ the equations above become

$$\begin{aligned} \dot{z}(t) &= \left(\mathcal{A} - \frac{\varphi(t)^2}{1-\varphi(t)^2 e^{\{n\}}(t)^2} \begin{bmatrix} \Gamma^{\{n\}} & 0_{1,n-1} \\ 0_{n-1,1} & 0_{n-1,n-1} \end{bmatrix} \right) z(t) \\ &\quad - \frac{\varphi(t)^2}{1-\varphi(t)^2 e^{\{n\}}(t)^2} \begin{bmatrix} \Gamma^{\{n\}} \\ 0_{n-1,1} \end{bmatrix} f(t), \qquad z(t_0) = 0, \\ e^{\{n\}}(t) &= \begin{bmatrix} 1 & 0_{1,n-1} \end{bmatrix} z(t) + f(t). \end{aligned} \tag{4.19}$$

The right hand side of this ordinary differential equation is defined on the open set

$$\mathcal{D} := \left\{ (t, z) \in [t_0, \infty) \times \mathbb{R}^n \;\middle|\; (t, z_1(t) + f(t)) \in \mathcal{F}_\varphi \right\},$$

with the performance funnel $\mathcal{F}_\varphi$ as in (1.3). It is readily verified that the right hand side of (4.19) satisfies a local Lipschitz condition with respect to $z(t)$ on the (relatively open) domain $\mathcal{D} \subset [t_0, \infty) \times \mathbb{R}^n$. Hence, by the standard theory of ordinary differential equations (see, e.g. [26, Thm. III.10.VI]), the initial-value problem (4.19) has a unique maximal solution

$$z^{\{n\}}(\cdot) \colon [t_0, \omega) \to \mathbb{R}^n, \qquad t_0 < \omega \leq \infty,$$

and moreover,

$$\operatorname{graph}(z^{\{n\}}) = \{(t, z^{\{n\}}(t)) |\; t \in [t_0, \omega)\} \subset \mathcal{D}$$

does not have compact closure in $\mathcal{D}$.

Now we show, that the solution $e^{\{n\}}$ does not approach the boundary of $\mathcal{D}$. Exploiting the Byrnes-Isidori structure of (4.12), we can represent $\mathfrak{D}^{\{n\}}$ in yet another way. Write $z^{\{n\}}(t) = \left(z_1^{\{n\}}(t), \tilde{z}^{\{n\}}(t) \right)$. Then eliminating $\tilde{z}^{\{n\}}$ from (4.12) by using the variation of constants formula yields that the solution z of (4.19) satisfies the integro-differential equation

$$\begin{aligned} \dot{z}_1(t) &= \mathcal{A}_{11}^{\{n\}} z_1(t) + \mathcal{A}_{12} \left(\int_0^t \mathrm{e}^{\mathcal{A}_{22}(t-\tau)} (\mathcal{A}_{12})^\top z_1(\tau) \,\mathrm{d}\tau \right) + \Gamma^{\{n\}} k_0(t, e^{\{n\}}(t)) e^{\{n\}}(t), \\ &= \left(\mathbf{T}^{\{n\}} z_1^{\{n\}} \right)(t) + \Gamma^{\{n\}} k_0(t, e^{\{n\}}(t)) e^{\{n\}}(t), \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} \mathbf{T}^{\{n\}} : L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}) &\to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}), \\ y &\mapsto \left(t \mapsto \mathcal{A}_{11}\, y(t) + \mathcal{A}_{12} \int_0^t \mathrm{e}^{\mathcal{A}_{22}(t-\tau)} \mathcal{A}_{12}^\top y(\tau) \,\mathrm{d}\tau \right). \end{aligned}$$

In order to prove that this solution is global we will exploit two crucial properties of the operator $\mathbf{T}^{\{n\}}$ which are proven in Appendix A. Firstly, $\mathbf{T}^{\{n\}}$ is negative semi-definite in the sense that

$$\forall\, t \geq 0,\ \forall e \in L^\infty([0, t]) \ :\ \int_0^t e(\tau)(\mathbf{T}^{\{n\}} e)(\tau) \,\mathrm{d}\tau \leq 0. \tag{4.21}$$

This follows from Lemma A.3 (ii), because $\mathcal{A}_{22}$ is a negative definite matrix. The second property is that

$$\|\dot{f}_0 - (\mathbf{T}^{\{n\}} f_0)\|_{L^\infty([0,\infty))} \leq \frac{\Gamma^{\{n\}}}{c_0} \cdot \|f_0\|_{W^{1,\infty}(\mathbb{R}_{\geq 0})}. \tag{4.22}$$

This holds because by Lemma A.3 (iii),

$$\begin{aligned}
&\|\dot f_0 - (\mathbf{T}^{\{n\}} f_0)\|_{L^\infty(\mathbb{R}_{geq0})} \\
&\qquad \le \lim_{s\to 0} \frac{1}{s} \left(\begin{bmatrix} 1, & 0_{1,n-1} \end{bmatrix} \begin{bmatrix} s - \mathcal{A}_{11} & -\mathcal{A}_{12} \\ -(\mathcal{A}_{12})^\top & sI - \mathcal{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0_{n-1,1} \end{bmatrix} \right)^{-1} \|f_0\|_{W^{1,\infty}([0,\infty))},
\end{aligned}$$

and with A and b as in (4.13), (4.15) we have the relation

$$\begin{aligned}
&\begin{bmatrix} 1, & 0_{1,n-1} \end{bmatrix} \begin{bmatrix} sI - \mathcal{A}_{11} & -\mathcal{A}_{12} \\ -(\mathcal{A}_{12})^\top & sI - \mathcal{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0_{n-1,1} \end{bmatrix} \\
&\quad = \begin{bmatrix} 1, & 0_{1,n-1} \end{bmatrix} (s - \mathcal{A})^{-1} \begin{bmatrix} \Gamma^{\{n\}} \\ 0_{n-1,1} \end{bmatrix} \frac{1}{\Gamma^{\{n\}}} \\
&\quad = b^\top (sI - A)^{-1} b \frac{1}{\Gamma^{\{n\}}} \\
&\quad = \sum_{k=0}^{n} \frac{c_k}{s + \lambda_k} \frac{1}{\Gamma^{\{n\}}}.
\end{aligned}$$

We use the representation (4.20) to show that the solution of (4.19) is global. Differentiating the last line of (4.19) shows for almost all $t \ge t_0$ that

$$\begin{aligned}
\dot e^{\{n\}}(t) &= \dot z_1(t) + \dot f(t) \\
&= (\mathbf{T}^{\{n\}} z_1)(t) + \Gamma^{\{n\}} k_0(t, e^{\{n\}}(t)) e^{\{n\}}(t) + \dot f_0(t) \\
&= (\mathbf{T}^{\{n\}} e^{\{n\}})(t) + (\dot f_0(t) - (\mathbf{T}^{\{n\}} f_0)(t)) + \Gamma^{\{n\}} k_0(t, e^{\{n\}}(t)) e^{\{n\}}(t).
\end{aligned} \tag{4.23}$$

Now define

$$\lambda := \inf_{t\in[t_0,\omega)} \varphi(t)^{-2}, \tag{4.24}$$

$$L := \text{ Lipschitz constant of } \varphi|_{[t_0,\infty)}(\cdot)^{-2}, \tag{4.25}$$

$$\hat\varphi := \max\left\{1, \sup_{t\in[t_0,\omega)} \varphi(t)^{-2}\right\}, \tag{4.26}$$

$$k(t) := k(t, e^{\{n\}}(t)) = \frac{\varphi(t)^2}{1 - (\varphi(t) e^{\{n\}}(t))^2}, \qquad t \in [t_0, \omega) \tag{4.27}$$

and

$$\varepsilon' := \min\left\{ \frac{\lambda}{2}, \lambda \left(\frac{4}{c_0} \|f_0\|_{W^{1,\infty}([0,\infty))} + \inf_{n\in\mathbb{N}} \frac{2L}{\Gamma^{\{n\}}} \right)^{-1}, \varphi(t_0)^{-2} - e^{\{n\}}(t_0)^2 \right\}. \tag{4.28}$$

We show that (4.16) holds for all $t \in [t_0, \omega)$. Seeking a contradiction, suppose that

$$\exists\, t_1 \in [t_0, \omega) \ : \ \varphi(t_1)^{-2} - e(t_1)^2 < \varepsilon'.$$

By continuity of φ and $e^{\{n\}}$, the maximum

$$t_{\varepsilon'} := \max\left\{ t \in [t_0, t_1) \,\middle|\, \varphi(t)^{-2} - (e^{\{n\}}(t))^2 = \varepsilon' \right\}$$

is attained and

$$\forall\, t \in (t_{\varepsilon'}, t_1) \ : \ \varphi(t)^{-2} - (e^{\{n\}}(t))^2 < \varepsilon'.$$

Therefore, the definitions (4.24) and (4.28) imply

$$\forall\, t \in (t_{\varepsilon'}, t_1) \;:\; e^{\{n\}}(t)^2 > \varphi(t)^{-2} - \varepsilon' \geq \lambda - \lambda/2 = \lambda/2\,. \tag{4.29}$$

Moreover, for all $t \in (t_{\varepsilon'}, t_1)$,

$$\frac{4\|f_0\|_{W^{1,\infty}([0,\infty))}}{\lambda c_0} + \frac{2L}{\Gamma^{\{n\}}\lambda} \overset{(4.28)}{\leq} \frac{1}{\varepsilon'} < \frac{1}{\varphi(t)^{-2} - (e^{\{n\}}(t))^2} \overset{(4.27)}{\leq} k(t),$$

and thus

$$\forall\, t \in (t_{\varepsilon'}, t_1) \;:\; \frac{2}{c_0}\|f_0\|_{W^{1,\infty}([0,\infty))} - \frac{\lambda k(t)}{2} \leq -\frac{L}{\Gamma^{\{n\}}}\,. \tag{4.30}$$

Finally, the application of butcher's hook to $\frac{\mathrm{d}}{\mathrm{d}t}(e^{\{n\}}(t))^2 = 2e^{\{n\}}(t)\dot{e}^{\{n\}}(t)$ and invoking (4.23) yields

$$\begin{aligned}
(e^{\{n\}}(t_1))^2 - (e^{\{n\}}(t_{\varepsilon'}))^2 &= \int_{t_{\varepsilon'}}^{t_1} \frac{\mathrm{d}}{\mathrm{d}\tau} e^{\{n\}}(\tau)^2 \,\mathrm{d}\tau = 2\int_{t_{\varepsilon'}}^{t_1} e^{\{n\}}(\tau)\dot{e}^{\{n\}}(\tau)\,\mathrm{d}\tau \\
= \quad & 2\int_{t_{\varepsilon'}}^{t_1} e^{\{n\}}(\tau)(\mathbf{T}^{\{n\}}e^{\{n\}})(\tau)e^{\{n\}}(\tau)\left(\dot{f}_0(\tau) - (\mathbf{T}^{\{n\}}e^{\{n\}})(\tau)\right) \\
& \quad - \Gamma^{\{n\}}k(\tau)\,(e^{\{n\}}(\tau))^2\,\mathrm{d}\tau \\
\overset{(4.21)}{\leq} \quad & 2\int_{t_{\varepsilon'}}^{t_1} |e^{\{n\}}(\tau)|\left\|\dot{f}(\tau) - (\mathbf{T}^{\{n\}}e^{\{n\}})(\tau)\right\|_\infty - \Gamma^{\{n\}}k(\tau)\,(e^{\{n\}}(\tau))^2\,\mathrm{d}\tau \\
\overset{(4.26),(4.22)}{\leq} \quad & 2\int_{t_{\varepsilon'}}^{t_1} \hat{\varphi}\frac{\Gamma^{\{n\}}}{c_0}\|f_0\|_{W^{1,\infty}[0,\infty)} - \Gamma^{\{n\}}k(\tau)\,(e^{\{n\}}(\tau))^2\,\mathrm{d}\tau \\
\overset{(4.29)}{\leq} \quad & \int_{t_{\varepsilon'}}^{t_1} \Gamma^{\{n\}}\left(\frac{2}{c_0}\|f_0\|_{W^{1,\infty}[0,\infty)} - \frac{\lambda k(t)}{2}\right)\mathrm{d}\tau \\
\overset{(4.30)}{\leq} \quad & \int_{t_{\varepsilon'}}^{t_1} -L\,\mathrm{d}\tau.
\end{aligned}$$

This implies

$$(e^{\{n\}}(t_1))^2 - (e^{\{n\}}(t_{\varepsilon'}))^2 \leq -L(t_1 - t_{\varepsilon'}) \overset{(4.25)}{\leq} -|\varphi(t_1)^{-2} - \varphi(t_{\varepsilon'})^{-2}|,$$

whence the contradiction

$$\varepsilon' = \varphi(t_{\varepsilon'})^{-2} - (e^{\{n\}}(t_{\varepsilon'}))^2 \leq \varphi(t_1)^{-2} - (e^{\{n\}}(t_1))^2 < \varepsilon'.$$

This proves (4.16) since ε' was chosen independently of n.

Finally, we show that $\omega = \infty$. Seeking a contradiction, suppose that $\omega < \infty$. Then the set

$$\mathcal{K} := \left\{(t, e) \in \mathcal{F}_\varphi \,\middle|\, t \in [t_0, \omega]\,,\ \varphi(t)^{-2} - |e|^2 \geq \varepsilon'\right\}$$

is a compact subset of $\mathcal{F}_\varphi$ with $(t, e^{\{n\}}(t)) \in \mathcal{K}$ for all $t \in [t_0, \omega)$ by (4.29). This contradicts the fact that the closure of graph $\left(e^{\{n\}}|_{[t_0,\omega)}\right)$ is not a compact set. Hence $\omega = \infty$. ∎

4.2. Step 2: Equicontinuity. We have shown in the previous section that (4.10) possesses for each $n \in \mathbb{N}$ a solution $e^{\{n\}}$. Further, these solutions are bounded away from the funnel boundary by a constant independent on n. We are now going use these findings to show that the set $\{e^{\{n\}} : n \in \mathbb{N}\}$ is equicontinuous. To this end we need the following estimate.

LEMMA 4.6. *Let Assumption 1 hold, define* $\mathfrak{D}^{\{n\}}$ *by* (4.11) *and let* $g \in L^\infty(\mathbb{R}_{\geq 0})$. *Then for all* $t_1, t_2 \in \mathbb{R}_{\geq 0}$ *and all* $n \in \mathbb{N}$ *holds*

$$|(\mathfrak{D}^{\{n\}}g)(t_1) - (\mathfrak{D}^{\{n\}}g)(t_2)| \leq \left(c_0|t_2 - t_1| + 2\sum_{k=1}^{\infty} \frac{c_k}{\lambda_k}\left(1 - \mathrm{e}^{-\lambda_k|t_2-t_1|}\right)\right) \cdot \|g\|_{L^\infty(\mathbb{R}_{\geq 0})}.$$

Proof. We assume without loss of generality that $t_1 \leq t_2$ and calculate

$$\begin{aligned}
&\left|(\mathfrak{D}^{\{n\}}g)(t_1) - (\mathfrak{D}^{\{n\}}g)(t_2)\right| \\
&= \left|\int_0^{t_1} h_n(t_1-\tau)g(\tau)d\tau - \int_0^{t_2} h_n(t_2-\tau)g(\tau)\mathrm{d}\tau\right| \\
&\leq \left|\int_0^{t_1} (h_n(t_1-\tau) - h_n(t_2-\tau))g(\tau)\mathrm{d}\tau\right| + \left|\int_{t_1}^{t_2} h_n(t_2-\tau)g(\tau)\mathrm{d}\tau\right| \\
&\leq \int_0^{t_1} |h_n(\tau) - h_n(t_2-t_1+\tau)|\,\mathrm{d}\tau \cdot \|g\|_{L^\infty(\mathbb{R}_{\geq 0})} + \int_0^{t_2-t_1} |h_n(\tau)|\,\mathrm{d}\tau \|g\|_{L^\infty(\mathbb{R}_{\geq 0})} \\
&\leq \left(\int_0^{t_1} \left|\sum_{k=0}^{n-1} c_k \mathrm{e}^{-\lambda_k\tau}(1-\mathrm{e}^{-\lambda_k(t_2-t_1)})\right| \mathrm{d}\tau + \int_0^{t_2-t_1} \sum_{k=0}^{n-1} c_k \mathrm{e}^{-\lambda_k\tau}\mathrm{d}\tau\right) \|g\|_{L^\infty(\mathbb{R}_{\geq 0})} \\
&= \left(\int_0^{t_1} \sum_{k=1}^{n-1} c_k \mathrm{e}^{-\lambda_k\tau}(1-\mathrm{e}^{-\lambda_k(t_2-t_1)})\mathrm{d}\tau + \int_0^{t_2-t_1} \sum_{k=0}^{n-1} c_k \mathrm{e}^{-\lambda_k\tau}\mathrm{d}\tau\right) \|g\|_{L^\infty(\mathbb{R}_{\geq 0})} \\
&= \Bigg(\sum_{k=1}^{n-1} \frac{c_k}{\lambda_k}(1-\mathrm{e}^{-\lambda_k t_1})(1-\mathrm{e}^{-\lambda_k(t_2-t_1)}) \\
&\qquad\qquad + c_0(t_2-t_1) + \sum_{k=1}^{n-1} \frac{c_k}{\lambda_k}(1-\mathrm{e}^{-\lambda_k(t_2-t_1)})\Bigg) \|g\|_{L^\infty(\mathbb{R}_{\geq 0})} \\
&\leq \left(2\sum_{k=1}^{\infty} \frac{c_k}{\lambda_k}(1-\mathrm{e}^{-\lambda_k(t_2-t_1)}) + c_0(t_2-t_1)\right) \|g\|_{L^\infty(\mathbb{R}_{\geq 0})}.
\end{aligned}$$

□

PROPOSITION 4.7. *The set of solutions* $\left\{ e^{\{n\}} \;\middle|\; n \in \mathbb{N} \right\}$ *to equation* (4.10) *that are given by Theorem 4.5, is uniformly equicontinuous. That is,*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall n \in \mathbb{N}\ \forall t_1, t_2 \in [t_0, \infty): \quad |t_1 - t_2| < \delta \Rightarrow |f(t_1) - f(t_2)| < \varepsilon.$$

Proof. Define the input signal corresponding to $e^{\{n\}}$ by

$$u^{\{n\}}(t) := \begin{cases} -\frac{\varphi(t)^2}{1-(\varphi(t)e^{\{n\}}(t))^2} e^{\{n\}}(t), & t \in [t_0, \infty), \\ 0, & t \in [0, t_0), \end{cases} \tag{4.31}$$

so that (4.10) reads

$$e^{\{n\}}(t) = (\mathfrak{D}^{\{n\}} u^{\{n\}})(t) \qquad \forall t \in [t_0, \infty).$$

Then the uniform estimate (4.16) in Theorem 4.5 implies that there is a $C > 0$ with $\|u^{\{n\}}\|_{L^\infty([t_0,\infty))} < C$ for all $n \in \mathbb{N}$. By Assumption 1 d) there exists some $N \in \mathbb{N}$ with

$$\sum_{k=N+1}^{\infty} \frac{c_k}{\lambda_k} < \frac{\varepsilon}{8C}.$$

Since $f_0 \in W^{1,\infty}(\mathbb{R}_{\geq 0})$ is uniformly continuous we may choose $\delta \in (0, \frac{\varepsilon}{4c_0 C})$ such that

$$|f_0(t_1) - f_0(t_1)| < \frac{\varepsilon}{4} \quad \text{for all } t_1, t_2 \text{ with } |t_1 - t_2| < \delta,$$

and

$$\sum_{k=1}^{N} \frac{c_k}{\lambda_k}(1 - \mathrm{e}^{-\lambda_k \delta}) < \frac{\varepsilon}{8C}.$$

For all $t_1, t_2 \in [t_0, \infty)$ with $|t_1 - t_2| < \delta$ we obtain by using Lemma 4.6

$$\begin{aligned}
&|e^{\{n\}}(t_1) - e^{\{n\}}(t_2)| \\
&\quad = |(\mathfrak{D}^{\{n\}} u^{\{n\}})(t_1) + f_0(t_1) - ((\mathfrak{D}^{\{n\}} u^{\{n\}})(t_2) + f_0(t_2))| \\
&\quad \leq \underbrace{|f_0(t_1) - f_0(t_2)|}_{<\frac{\varepsilon}{4}} + \left|(\mathfrak{D}^{\{n\}} u^{\{n\}})(t_1) - (\mathfrak{D}^{\{n\}} u^{\{n\}})(t_2)\right| \\
&\quad \leq \frac{\varepsilon}{4} + \left(c_0|t_1 - t_2| + 2\sum_{k=1}^{\infty} \frac{c_k}{\lambda_k}(1 - \mathrm{e}^{-\lambda_k \delta})\right) \cdot \underbrace{\|u^{\{n\}}\|_{L^\infty(\mathbb{R}_{\geq 0})}}_{<C} \\
&\quad \leq \frac{\varepsilon}{4} + \Bigg(\underbrace{c_0 \delta}_{<\frac{\varepsilon}{4C}} + \underbrace{2\sum_{k=1}^{N} \frac{c_k}{\lambda_k}(1 - \mathrm{e}^{-\lambda_k \delta})}_{\leq \frac{\varepsilon}{4C}} + \underbrace{2\sum_{k=N+1}^{\infty} \frac{c_k}{\lambda_k}}_{\leq \frac{\varepsilon}{4C}}\Bigg) \cdot C < \varepsilon.
\end{aligned}$$

□

4.3. Step 3: Convergence and existence of a solution.

LEMMA 4.8. *For $p \in \{2, \infty\}$, the* truncated mapping $\mathfrak{D}^{\{n\}}$ *defined in* (4.11) *fulfills*

$$\|\mathfrak{D}|_{[0,t]} - \mathfrak{D}^{\{n\}}|_{[0,t]}\|_{\mathcal{B}(L^p([0,t]))} \leq \sum_{k=n}^{\infty} \frac{c_k}{\lambda_k}. \tag{4.32}$$

Proof. Let $p \in \{2, \infty\}$ and $u \in L^p([0,t])$. Then, by definition of $\mathfrak{D}^{\{n\}}$, there holds

$$(\mathfrak{D} - \mathfrak{D}^{\{n\}})u = \left(t \mapsto \int_0^t (h(t-\tau) - h_n(t-\tau))\, u(\tau) \mathrm{d}\tau\right),$$

where $h(\cdot) - h_n(\cdot) = \sum_{k=n+1}^{\infty} c_k \mathrm{e}^{-\lambda_k \cdot}$. With the norm bound

$$\|h - h_n\|_{L^1([0,t])} = \sum_{k=n+1}^{\infty} \frac{c_k}{\lambda_k}(1 - \mathrm{e}^{-\lambda_k t}) \leq \sum_{k=n+1}^{\infty} \frac{c_k}{\lambda_k},$$

the desired result follows from Young's inequality [6, Thm. 3.9.4]. □

We finally come to the proof of Theorem 4.2.

Proof. [Proof of Theorem 4.2] Let $\left\{ e^{\{n\}} \;\middle|\; n \in \mathbb{N} \right\}$ be the set of solutions of (4.10) from Theorem 4.5. More precisely, we assume that for each $n \in \mathbb{N}$, the function $e^{\{n\}} \in \mathcal{C}(\mathbb{R}_{\geq 0})$ satisfies the augmented equation (4.17), which means in particular that $e^{\{n\}}|_{[0,t_0]} = f_0|_{[0,t_0]}$ for all n. Let $t \in \mathbb{R}_{\geq 0}$ be arbitrary. Since the sequence $(e^{\{n\}}|_{[0,t]})_{n\in\mathbb{N}}$ is bounded by $1/\|\varphi\|_{L^\infty([t_0,\infty))}$ and, by Proposition 4.7, equicontinuous, we can conclude from the Arzelà-Ascoli Theorem [23, Thm. 11.28] that $(e^{\{n\}}|_{[0,t]})_{n\in\mathbb{N}}$ contains a convergent subsequence $(e^{\{n_k\}}|_{[0,t]})_{k\in\mathbb{N}}$.

Let $e \in \mathcal{C}([0,t])$ be the limit of this subsequences, i.e.

$$\lim_{k\to\infty} \|e - (\mathrm{e}^{\{n_k\}})|_{[0,t]}\|_{L^\infty([0,t))} = 0.$$

Since by (4.16), the function $e^{\{n\}}$ stays away from the funnel boundary, so does e. I.e. (4.7) holds. Hence, there is some $\delta > 0$ such that $\|\varphi^2 e^2\|_{L^\infty([0,t])} \leq 1 - \delta$, which is why the inputs u and $u^{\{n\}}$ defined by

$$u(t) := \begin{cases} -\frac{\varphi(t)^2}{1-(\varphi(t)e(t))^2} e(t), & t \in [t_0, \infty), \\ 0, & t \in [0, t_0) \end{cases}$$

and (4.31) respectively, are well-defined and satisfy

$$\begin{aligned}
&\|u - u^{\{n_k\}}\|_{L^\infty([0,t])} \\
&\quad = \left\| \frac{\varphi^2(e - e^{\{n_k\}}) - \varphi^4 e e^{\{n_k\}}(e - e^{\{n_k\}})}{(1-\varphi^2 e^2)(1-\varphi^2(e^{\{n_k\}})^2)} \right\|_{L^\infty([t_0,t])} \\
&\quad \leq \frac{1}{\delta^2}\Big(\|\varphi\|^2_{L^\infty([t_0,t])} + \|\varphi\|^4_{L^\infty([t_0,t])} \|e\|_{L^\infty([t_0,t])} \|e^{\{n_k\}}\|_{L^\infty([t_0,t])} \Big) \|e - e^{\{n_k\}}\|_{L^\infty([t_0,t])}.
\end{aligned}$$

For $k \to \infty$ this implies

$$\lim_{k\to\infty} \|u - u^{\{n_k\}}|_{[0,t]}\|_{L^\infty([0,t])} = 0.$$

Recall that Lemma 4.8 shows

$$\lim_{k\to\infty} \|\mathfrak{D} - \mathfrak{D}^{\{n_k\}}\|_{\mathcal{B}(L^\infty([0,t]))} = 0.$$

Therefore, in the equation

$$\begin{aligned}
&\|e - (\mathfrak{D}(u) + f)\|_{L^\infty([t_0,t])} \\
&\qquad = \|(e - e^{\{n_k\}}) - ((\mathfrak{D}(u) + f_0) - (\mathfrak{D}^{\{n_k\}}(u^{\{n_k\}}) + f_0))\|_{L^\infty([0,t])} \\
&\qquad \leq \|e - e^{\{n_k\}}\|_{L^\infty([0,t])} + \|\mathfrak{D}(u) - \mathfrak{D}^{\{n_k\}}(u^{\{n_k\}})\|_{L^\infty([0,t])} \\
&\qquad \leq \|e - e^{\{n_k\}}\|_{L^\infty([0,t])} + \|(\mathfrak{D} - \mathfrak{D}^{\{n_k\}})(u) + \mathfrak{D}^{\{n_k\}}(u - u^{\{n_k\}})\|_{L^\infty([0,t])} \\
&\qquad \leq \|e - e^{\{n_k\}}\|_{L^\infty([0,t])} + \|\mathfrak{D} - \mathfrak{D}^{\{n_k\}}\|_{\mathcal{B}(L^\infty([0,t]))} \cdot \|u\|_{L^\infty([0,t])} \\
&\qquad\qquad + \|\mathfrak{D}^{\{n_k\}}\|_{\mathcal{B}(L^\infty([0,t]))} \|(u - u^{\{n_k\}})\|_{L^\infty([0,t])},
\end{aligned}$$

the right hand side tends to zero as $k \to \infty$. This proves that the function e satisfies (4.6a) on $[t_0, t]$. Since this construction was done with arbitrary $t \in [t_0, \infty)$, it enables us to construct a function $e : [t_0, \infty) \to \mathbb{R}$ that fulfills all the claims of the theorem. Finally, the uniform continuity of e is a consequence of the fact that e satisfies the convolution equation (4.6a) and that the convolution of $h \in L^1(\mathbb{R}_{\geq 0})$ and $u \in L^\infty([t_0, \infty))$ is bounded and uniformly continuous according to [12, Chap. 2, Thm. 2.2]. ∎

5. The heat equation with funnel control. With the results of the previous section, we can now prove that the funnel controller applied to the heat equation (1.1) yields a global solution to this equation, such that the error between the reference and output signal evolves in the performance funnel.

5.1. Existence of a solution. In a first step, we construct a (mild) solution of the system (3.1) in the sense of well-posed linear systems, cf. (3.7), and analyze the input and output signals. An analysis of the state space trajectory will follow in the next section.

THEOREM 5.1. *Let $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0})$, $x_0 \in L^2(\Omega; \mathbb{R})$ be given. Pick any $\varphi \in \Phi$ and define the funnel feedback gain function k by* (4.6b)*. Then there exists a unique function $x \in \mathcal{C}([0, \infty), L^2(\Omega))$ such that, with the systems operators $\mathfrak{A}$, $\mathfrak{B}_t$, $\mathfrak{C}$, $\mathfrak{D}$ defined in* (3.5)*, the equations*

$$x(t) = \mathfrak{A}(t)x_0 + \mathfrak{B}_t u|_{[0,t]}, \tag{5.1a}$$

$$y(t) = \mathfrak{C}_t x_0 + \mathfrak{D}_t u, \tag{5.1b}$$

$$u(t) = k(t, y(t) - y_{\mathrm{ref}}(t))(y(t) - y_{\mathrm{ref}}(t)) \tag{5.1c}$$

hold for all $t \in \mathbb{R}_{>0}$. Moreover,

(i) the input fulfills $u \in \mathcal{BUC}(\mathbb{R}_{>0})$;

(ii) the output function satisfies $y \in \mathcal{C}(\mathbb{R}_{>0})$ and $y|_{[\delta,\infty)} \in \mathcal{BUC}([\delta, \infty))$ for all $\delta > 0$;

(iii) the tracking error $e := y - y_{\mathrm{ref}}$ evolves within the funnel $\mathcal{F}_\varphi$ with uniform distance to the funnel boundary in the sense that (2.1) *holds.*

Proof. By Lemma 3.8, the input-output map $\mathfrak{D}$ has a representation

$$\mathfrak{D}u = \left(t \mapsto \int_0^t h(\tau)u(t-\tau)\mathrm{d}\tau \right) \quad \forall u \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0}),$$

with $h(\cdot)$ defined as in (4.2) and fulfilling Assumption 1, see Lemma 3.7. Let $\gamma_0 > 0$ be such that $\varphi \in \Phi_{\gamma_0}$. By Lemma 3.3 we have $\mathfrak{C}x_0|_{[\gamma_0,\infty)} \in W^{1,\infty}([\gamma_0, \infty))$, which together with $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0})$ implies that the function

$$f := \mathfrak{C}x_0 - y_{\mathrm{ref}}$$

fulfills $f|_{[\gamma_0,\infty)} \in W^{1,\infty}([\gamma_0, \infty))$. Thus, by Corollary 4.3, there exists a solution $e \in \mathcal{BUC}([\gamma_0, \infty))$ of the Volterra equation (4.8) with f as above. The corollary also states that

$$\exists\, \varepsilon' > 0\ \forall t > \gamma_0 \ :\ |e(t)|^2 \leq \varphi(t)^{-2} - \varepsilon'.$$

Define the function

$$u(t) := \begin{cases} 0, & t \in [0, \gamma_0), \\ k(t, e(t)) \cdot e(t), & t \geq \gamma_0. \end{cases}$$

The estimate above and the definition of k imply that the function $t \mapsto k(t, e(t))$ is bounded. Hence, u is bounded and a short calculation using the boundedness of k and the uniform continuity of e on $[\gamma_0, \infty)$ shows that u is uniformly continuous on $\mathbb{R}_{>0}$. So (i) is proven.

With this u we define the function x via (5.1a) and y via

$$y(t) := \begin{cases} \mathfrak{C}_t x_0, & t \in [0, \gamma_0), \\ e(t) + y_{\mathrm{ref}}(t), & t \geq \gamma_0. \end{cases}$$

Then y is continuous and its restriction to $[\gamma_0, \infty)$ is in $\mathcal{BUC}([\gamma_0, \infty))$ since e and y_{ref} are. This implies (ii) because the uniform continuity on any compact interval $[\delta, \gamma_0]$ is trivial.

Extending e to $\mathbb{R}_{>0}$ by $e := y - y_{\mathrm{ref}}$, we get

$$\varphi(t)^2 e(t)^2 \leq 1 - \varphi(t)^2 \cdot \varepsilon' \quad \forall t > 0$$

because $\varphi|_{(0,\gamma_0)} = 0$. Due to the continuity of e at γ_0 and the definition of Φ_{γ_0} this implies for a suitable ε

$$\varphi(t)^2 e(t)^2 \leq 1 - \varepsilon \quad \forall t > 0,$$

so the assertion (iii) holds.

We check that all the equations in (5.1) hold. Since $u|_{(0,\gamma_0)}$ is zero, the definition of y immediately gives (5.1b) for $t \in (0, \gamma_0)$. For $t \geq \gamma_0$, we obtain the definition of f that

$$y(t) = e(t) + y_{\mathrm{ref}}(t) \overset{(4.8)}{=} \mathfrak{D}_t u + f(t) = \mathfrak{D}_t u + \mathfrak{C} x_0 - y_{\mathrm{ref}}(t) + y_{\mathrm{ref}}(t) = \mathfrak{C} x_0 + \mathfrak{D}_t u.$$

Hence, (5.1b) holds everywhere. Equation (5.1a) is fulfilled by the definition of x and (5.1c) is fulfilled by the definition of u and the fact that $k = 0$ for $t \in [0, \gamma_0)$. Finally, the uniqueness of these solutions follows from the uniqueness of the solution in Corollary 4.3. ∎

5.2. Boundedness and regularity of the solution. Note that Theorem 5.1 does not yet say anything about the norm of the solution x. In this section we will show that x is bounded in the norm of the state space $L^2(\Omega)$. To do this, we will exploit the fact that any constant output feedback stabilizes the system exponentially.

Well-posedness of regular infinite-dimensional systems under output feedback is well understood, see Weiss in [28]. By the results in [22] the heat equation (1.1) with output feedback $u(t) = v(t) - ky(t)$ defines a well-posed linear system.

Lemma 5.2 ([22, Thm. 6.3 & Thm. 6.4]). *Under the assumptions of Theorem 5.1, let y and u be the input and output functions defined in* (5.1) *and let K be any positive constant. If we set $v(t) := u(t) + Ky(t)$ for all $t > 0$, then the state x defined in* (5.1a) *satisfies*

$$x(t) = \mathfrak{A}_K(t) x_0 + \mathfrak{B}_{K,t} v, \qquad t > 0, \tag{5.2}$$

where $\mathfrak{A}_K$ is an exponentially stable, analytic semigroup on $L^2(\Omega)$ generated by the self-adjoint, negative operator

$$A_K x = \Delta x, \; D(A_K) = \left\{ x \in W^{2,2}(\Omega) \,\middle|\, \partial_\nu x(\zeta) = -K \int_{\partial\Omega} x(\xi) \mathrm{d}\sigma_\xi \quad \forall \zeta \in \partial\Omega \right\} \tag{5.3}$$

and

$$\mathfrak{B}_{k,t}v = \int_0^t \mathfrak{A}_K(t-\tau)|_{D(A_K^*)'} Bv(\tau)\,\mathrm{d}\tau \qquad \text{in } D(A_K^*)'.$$

Here, $\mathfrak{A}_K(t)|_{D(A_K^*)'}$ is the extension of $\mathfrak{A}_K(t)$ to $D(A_K^*)'$. In particular, the range of B is contained in this space.

PROPOSITION 5.3. *The solution in Theorem 5.1 satisfies*

$$\sup_{t\geq 0} \|x(t)\|_{L^2(\Omega)} < \infty, \tag{5.4}$$

$x \in \mathcal{C}(\mathbb{R}_{>0}; W^{1,2}(\Omega))$ and for some $\omega, c > 0$

$$\|x(t)\|_{W^{1,2}(\Omega)} < c\left(1 + t^{-\frac{1+\theta}{2}} e^{-\omega t}\right) \qquad \forall t \in \mathbb{R}_{>0}. \tag{5.5}$$

If $x_0 \in W^{1,2}(\Omega)$, then $x \in \mathcal{C}(\mathbb{R}_{\geq 0}; W^{1,2}(\Omega))$ and

$$\sup_{t\geq 0} \|x(t)\|_{W^{1,2}(\Omega)} < \infty. \tag{5.6}$$

Proof. Choose any $K > 0$ and define $v(t) := u(t) + Ky(t) \in L^\infty(\mathbb{R}_{\geq 0})$. Then by Lemma 5.2 the function x satisfies

$$x(t) = \mathfrak{A}_K(t)x_0 + \mathfrak{B}_{K,t}v. \tag{5.7}$$

We use Lemma B.3 to show that $\mathfrak{A}_K$ regularizes the solution x. Pick some $\theta \in (\frac{1}{2}, 1)$ then $\operatorname{ran} B \subset W^{\theta,2}(\Omega)'$ because B^* is well-defined and continuous from $W^{\theta,2}(\Omega)$ into $\mathbb{C}$, see Section 3. So Lemma B.3 implies that $\mathfrak{A}_K(t-\tau)|_{W^{\theta,2}(\Omega)'} Bv(\tau)$ is in $W^{1,2}(\Omega)$ and

$$\left\|\mathfrak{A}_K(t-\tau)|_{W^{\theta,2}(\Omega)'} Bv(\tau)\right\|_{W^{1,2}(\Omega)} \leq c\left(1 + (t-\tau)^{-\frac{1+\theta}{2}}\right) e^{-\omega(t-\tau)} \|B\| \|v\|_\infty.$$

Since the real-valued function on the right hand side is integrable over $[0,t)$, the integral in $\mathfrak{B}_{K,t}v$ converges in $W^{1,2}(\Omega)$ and

$$\begin{aligned}
\|\mathfrak{B}_{K,t}v\|_{W^{1,2}(\Omega)} &\leq c\int_0^t e^{-\omega(t-\tau)} + (t-\tau)^{-\frac{1+\theta}{2}} e^{-\omega(t-\tau)} \mathrm{d}\tau \cdot \|B\|\|v\|_\infty \\
&= c\|B\|\|v\|_\infty \int_0^t e^{-\omega\tau} + \tau^{-\frac{1+\theta}{2}} e^{-\omega\tau} \mathrm{d}\tau \\
&= c\|B\|\|v\|_\infty \left(\frac{1-e^{-\omega t}}{\omega} + \int_0^1 \tau^{-\frac{1+\theta}{2}} e^{-\omega\tau} \mathrm{d}\tau + \int_1^\infty \tau^{-\frac{1+\theta}{2}} e^{-\omega\tau} \mathrm{d}\tau\right) \\
&\leq c\|B\|\|v\|_\infty \left(\frac{1-e^{-\omega t}}{\omega} + \int_0^1 \tau^{-\frac{1+\theta}{2}} \mathrm{d}\tau + \int_1^\infty e^{-\omega\tau} \mathrm{d}\tau\right) \\
&= c\|B\|\|v\|_\infty \left(\frac{1-e^{-\omega t}}{\omega} + \frac{2}{1-\theta} + \frac{e^{-\omega}}{\omega}\right).
\end{aligned}$$

This shows

$$\sup_{t>0} \|\mathfrak{B}_{K,t}v\|_{W^{1,2}(\Omega)} < \infty. \tag{5.8}$$

For $t \in \mathbb{R}_{\geq 0}$ we get

$$
\begin{aligned}
&\|\mathfrak{B}_{K,t+h}v - \mathfrak{B}_{K,t}v\|_{W^{1,2}(\Omega)} \\
&\quad = \left\| \int_0^{t+h} \mathfrak{A}(\tau)Bv(t+h-\tau)\,\mathrm{d}\tau + \int_0^t \mathfrak{A}(\tau)Bv(t-\tau)\mathrm{d}\,\tau \right\|_{W^{1,2}(\Omega)} \\
&\quad = \left\| \int_t^{t+h} \mathfrak{A}(\tau)Bv(t+h-\tau)\,\mathrm{d}\tau + \int_0^t \mathfrak{A}(\tau)B(v(t+h-\tau)-v(t-\tau))\,\mathrm{d}\tau \right\|_{W^{1,2}(\Omega)} \\
&\quad \leq \int_t^{t+h} \|\mathfrak{A}(\tau)Bv(t+h-\tau)\|_{W^{1,2}(\Omega)}\,\mathrm{d}\tau \\
&\qquad\qquad + \int_0^t \|\mathfrak{A}(\tau)B(v(t+h-\tau)-v(t-\tau))\|_{W^{1,2}(\Omega)}\,\mathrm{d}\tau \\
&\quad \leq c\|B\|\|v\|_\infty \int_t^{t+h} 1+\tau^{-\frac{1+\theta}{2}}\,\mathrm{d}\tau \\
&\qquad\qquad + c\|B\| \int_0^t 1+\tau^{-\frac{1+\theta}{2}}\,\mathrm{d}\tau \sup_{\tau\in[0,t+h]} \|v(t+h-\tau)-v(t-\tau)| \\
&\quad \leq c\|B\|\|v\|_\infty \int_t^{t+h} 1+\tau^{-\frac{1+\theta}{2}}\,\mathrm{d}\tau \\
&\qquad\qquad + c\|B\| \int_0^t 1+\tau^{-\frac{1+\theta}{2}}\,\mathrm{d}\tau \sup_{\tau\in[0,t+h]} \|v(t+h-\tau)-v(t-\tau)| \\
&\quad \to 0 \quad (h \to 0),
\end{aligned}
$$

because v is uniformly continuous and the function $1+\tau^{-\frac{1+\theta}{2}}$ is integrable on the compact interval $[0, t+h]$. This proves that, on $\mathbb{R}_{\geq 0}$, the mapping $t \mapsto \mathfrak{B}_{K,t}v$ is continuous with respect to the $W^{1,2}(\Omega)$ norm.

Let us first assume that $x_0 \in W^{1,2}(\Omega)$. Since $\mathfrak{A}_K$ restricts to a bounded, strongly continuous semigroup on $D((-A_K)^{\frac{1}{2}}) = W^{1,2}(\Omega)$, the mapping $t \mapsto \mathfrak{A}(t)x_0$ is continuous and bounded with respect to the $W^{1,2}(\Omega)$ norm. Therefore the above calculations and equation (5.7) show that $x \in \mathcal{C}(\mathbb{R}_{\geq 0}; W^{1,2}(\Omega))$ and the bound (5.6) holds.

Now for general $x_0 \in L^2(\Omega)$, Lemma 3.3 states that $\mathfrak{A}(\delta)x_0 \in W^{1,2}(\Omega)$ for arbitrary $\delta > 0$, whence the argumentation from above shows $x \in \mathcal{C}(\mathbb{R}_{>0}; W^{1,2}(\Omega))$. Finally, the norm bounds (5.4) and(5.5) are consequences of (5.8) together with

$$\|\mathfrak{A}_K(t)x_0\|_{W^{1,2}(\Omega)} \leq c\left(1+t^{-\frac{1}{2}}\right)e^{-\omega t}$$

and

$$\|\mathfrak{A}_K(t)x_0\|_{L^2(\Omega)} \leq e^{-\omega t}.$$

□

5.3. Proof of Theorem 2.2. The following proof of Theorem 2.2 is mostly a summary of the previous result.

Proof. Let u, y and x be the functions in Theorem 5.1 and define the error $e := y - y_{\mathrm{ref}}$. Then Theorem 5.1 already contains the statements (ii), (iii) and (iv) of Theorem 2.2. Proposition 5.3 shows the bounds on the state function x that are claimed in Theorem 2.2 (i).

It remains to prove part (iv) of Theorem 2.2, i.e. that u, y and x fulfill the weak formulation of the partial differential equation. The state equation (5.1a) implies for all $\psi \in D(A)$ that

$$\begin{aligned}\langle x(t+h), \psi\rangle_{L^2(\Omega)} - \langle x(t), \psi\rangle_{L^2(\Omega)} &= \int_t^{t+h} \langle x(\tau), A^*\psi\rangle_{L^2(\Omega)} + \langle u(\tau), B^*\psi\rangle \,\mathrm{d}\tau \\ &= \int_t^{t+h} \langle x(\tau), \Delta\psi\rangle_{L^2(\Omega)} + u(\tau)\int_{\partial\Omega} \overline{\psi}\,\mathrm{d}\sigma_\xi\,\mathrm{d}\tau \\ &= -\int_t^{t+h} \langle \nabla x(\tau), \nabla\psi\rangle_{L^2(\Omega)} - u(\tau)\int_{\partial\Omega} \overline{\psi}\,\mathrm{d}\sigma_\xi\,\mathrm{d}\tau,\end{aligned}$$

see e.g. [24, Thm. 3.8.2 (i)]. Since $D(A)$ is dense in $W^{1,2}(\Omega)$ [22, Lem. 2.2 (iii)], this equation extends continuously to all $\psi \in W^{1,2}(\Omega)$. We divide the equation by $h > 0$ and let h tend to zero. Then the continuity of x with respect to the $W^{1,2}(\Omega)$ norm and the continuity of u yield

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\langle x(t), \psi\rangle_{L^2(\Omega)} = -\langle \nabla x(t), \nabla\psi\rangle_{L^2(\Omega)} + u(t)\int_{\partial\Omega} \overline{\psi}\,\mathrm{d}\sigma_\xi \qquad \forall \psi \in W^{1,2}(\Omega).$$

This completes the proof of Theorem 2.2. □

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Appendix A. Convolution operators for symmetric systems.

LEMMA A.1. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric and negative definite, let $b \in \mathbb{R}^n$. Then the operator $T_{A,b} : L^\infty(\mathbb{R}_{\geq 0}) \to L^\infty(\mathbb{R}_{\geq 0})$ with*

$$T_{A,b}u = \left(t \mapsto \int_0^t b^\top \mathrm{e}^{A(t-\tau)} b u(\tau)\mathrm{d}\tau\right) \qquad \forall u \in L^\infty(\mathbb{R}_{\geq 0})$$

is bounded with $\|T_{A,b}\|_{\mathcal{B}(L^\infty(\mathbb{R}_{\geq 0}))} = -b^\top A^{-1} b$. Moreover, for all $u \in L^\infty(\mathbb{R}_{\geq 0})$ and $t \in \mathbb{R}_{\geq 0}$, there holds

$$\int_0^t u(\tau)(T_{A,b}u)(\tau)\mathrm{d}\tau \geq 0. \tag{A.1}$$

Proof. Consider the function $h_{A,b} = (t \mapsto b^\top \mathrm{e}^{At} b) \in L^1(\mathbb{R}_{\geq 0})$. The symmetry of A implies that $h_{A,b}$ is a nonnegative function, and thus

$$\|h_{A,b}\|_{L^1(\mathbb{R}_{\geq 0})} = \int_0^\infty h_{A,b}(\tau)\mathrm{d}\tau = \int_0^\infty b^\top \mathrm{e}^{A\tau} b \mathrm{d}\tau = -b^\top A^{-1} b.$$

The operator $T_{A,b}$ represents convolution with $h_{A,b}$, whence we obtain by Young's inequality [6, Thm. 3.9.4] that $T_{A,b} \in \mathcal{B}(L^\infty(\mathbb{R}_{\geq 0}))$ with

$$\|T_{A,b}\|_{\mathcal{B}(L^\infty(\mathbb{R}_{\geq 0}))} \leq \|h_{A,b}\|_{L^1(\mathbb{R}_{\geq 0})} = -b^\top A^{-1} b.$$

By applying $u \equiv 1$ to $T_{A,b}$, we further see that $\|T_{A,b}\|_{\mathcal{B}(L^\infty(\mathbb{R}_{\geq 0}))} \geq -b^\top A^{-1} b$.

To see that (A.1) holds true, we make use of the fact that $y = T_{A,b}u$ is given by $b^\top x$, where x is the solution of the initial value problem

$$\dot{x}(t) = Ax(t) + bu(t), \qquad x(0) = 0.$$

Thereby, we get

$$\begin{aligned}0 \leq \frac{1}{2}\|x(t)\|^2 &= \int_0^t x(\tau)^\top \dot{x}(\tau)\mathrm{d}\tau = \int_0^t x(\tau)^\top Ax(\tau) + x(\tau)^\top bu(\tau)\mathrm{d}\tau \\ &= \int_0^t \underbrace{x(\tau)^\top Ax(\tau)}_{\leq 0} + \underbrace{(b^\top x(\tau))}_{=y(\tau)} u(\tau)\mathrm{d}\tau \leq \int_0^t y(\tau)u(\tau)\mathrm{d}\tau = \int_0^t u(\tau)(T_{A,b}u)(\tau)\mathrm{d}\tau.\end{aligned}$$

□

Remark A.2. *We note that property* (A.1) *is called* passivity *of a system [30].*

Lemma A.3. *Let $A_{22} \in \mathbb{R}^{n-1\times n-1}$ be symmetric and negative definite, and let $A_{12} \in \mathbb{R}^{1\times n-1}$, $A_{11} \in \mathbb{R}$ such that the matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{bmatrix}$$

is singular and negative semi-definite. Then the following holds true:

(i) $A_{11} = A_{12}A_{22}^{-1}A_{12}^\top$*;*

(ii) for all $t \in \mathbb{R}_{\geq 0}$, the operator $\mathbf{T} : L^\infty([0,t]) \to L^\infty([0,t])$ with

$$\mathbf{T}x = \left(t \mapsto A_{11}x(t) + \int_0^t A_{12}\mathrm{e}^{A_{22}(t-\tau)}A_{12}^\top x(\tau)\,\mathrm{d}\tau\right) \qquad \forall x \in L^\infty([0,t])$$

fulfills

$$\int_0^t x(\tau)(\mathbf{T}x)(\tau)\,\mathrm{d}\tau \leq 0.$$

(iii) The operator $T : W_0^{1,\infty}(\mathbb{R}_{\geq 0}) \to L^\infty(\mathbb{R}_{\geq 0})$

$$Ty = \left(t \mapsto \dot{y}(t) + \int_0^t A_{12}\mathrm{e}^{A_{22}(t-\tau)}A_{12}^\top y(\tau)\,\mathrm{d}\tau + A_{11}y(t)\right) \qquad \forall y \in L^\infty([0,t])$$

fulfills

$$\begin{aligned}&\|T\|_{\mathcal{B}(W_0^{1,\infty}(\mathbb{R}_{\geq 0}),L^\infty(\mathbb{R}_{\geq 0}))} \\ &\leq \lim_{s\to 0}\frac{1}{s}\left(\begin{bmatrix}1 & 0_{1,n-1}\end{bmatrix}\begin{bmatrix} sI - A_{11} & -A_{12} \\ -A_{12}^\top & sI - A_{22}\end{bmatrix}^{-1}\begin{bmatrix}1 \\ 0_{n-1,1}\end{bmatrix}\right)^{-1}.\end{aligned}$$

Proof.

(i) By using elementary row transformations and the singularity of A, we obtain

$$0 = \det\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22}\end{bmatrix} = \det(A_{22})\cdot(A_{11} - A_{12}A_{22}^{-1}A_{12}^\top).$$

Then the result follows from $\det(A_{22}) \neq 0$, which holds true since A_{22} is negative definite.

(ii) By using the Cauchy-Schwarz and Young's inequality [6, Thm. 3.9.4], we obtain

$$\begin{aligned}&\left|\int_0^t x(\tau_1)\int_0^{\tau_1} A_{12}\mathrm{e}^{A_{22}(\tau_1-\tau)}A_{12}^\top x(\tau)\,\mathrm{d}\tau\,\mathrm{d}\tau_1\right| \\ &\leq \|x\|_{L^2([0,t])}\cdot\left\|\int_0^{\cdot} A_{12}\mathrm{e}^{A_{22}(\cdot-\tau)}A_{12}^\top x(\tau)\,\mathrm{d}\tau\right\|_{L^2([0,t])} \\ &\leq \|x\|_{L^2([0,t])}^2\cdot\|A_{12}\mathrm{e}^{A_{22}\cdot}A_{12}^\top\|_{L^1([0,t])} \\ &\leq \|x\|_{L^2([0,t])}^2\cdot\|A_{12}\mathrm{e}^{A_{22}\cdot}A_{12}^\top\|_{L^1([0,\infty))} \leq \|x\|_{L^2([0,t])}^2\cdot(-A_{12}A_{22}^{-1}A_{12}^\top).\end{aligned}$$

This gives rise to the estimate

$$\begin{aligned}\int_0^t x(\tau)(\mathbf{T}x)(\tau)\,\mathrm{d}\tau =& A_{11}\|x\|^2_{L^2([0,t])} + \int_0^t x(\tau_1)\int_0^{\tau_1} A_{12}\mathrm{e}^{A_{22}(\tau_1-\tau)}A_{12}^\top x(\tau)\,\mathrm{d}\tau\,\mathrm{d}\tau_1\\ &\le A_{11}\|x\|^2_{L^2([0,t])} - A_{12}A_{22}^{-1}A_{12}^\top\|x\|^2_{L^2([0,t])} \overset{\text{(i)}}{=} 0.\end{aligned}$$

(iii) Let $y \in W_0^{1,\infty}$. Then by integration by parts we obtain

$$\begin{aligned}&\int_0^t A_{12}\mathrm{e}^{A_{22}(t-\tau)}A_{12}^\top y(\tau)\,\mathrm{d}\tau\\ =& A_{12}\mathrm{e}^{A_{22}t}\int_0^t \mathrm{e}^{-A_{22}\tau}A_{12}^\top y(\tau)\,\mathrm{d}\tau\\ =& A_{12}\mathrm{e}^{A_{22}t}\left(-A_{22}^{-1}\mathrm{e}^{-A_{22}\tau}A_{12}^\top y(\tau)\big|_{\tau=0}^{\tau=t} + A_{22}^{-1}\int_0^t \mathrm{e}^{-A_{22}\tau}A_{12}^\top\dot{y}(\tau)\,\mathrm{d}\tau\right)\\ =& -A_{12}\mathrm{e}^{A_{22}t}A_{22}^{-1}\mathrm{e}^{-A_{22}t}A_{12}^\top y(t) + A_{12}\mathrm{e}^{A_{22}t}A_{22}^{-1}\int_0^t \mathrm{e}^{-A_{22}\tau}A_{12}^\top\dot{y}(\tau)\,\mathrm{d}\tau\\ =& -A_{12}A_{22}^{-1}A_{12}^\top y(t) - A_{12}\int_0^t \mathrm{e}^{-A_{22}(t-\tau)}A_{12}^\top\dot{y}(\tau)\,\mathrm{d}\tau\\ \overset{(i)}{=}& -A_{11}y(t) - A_{12}\int_0^t \mathrm{e}^{-A_{22}(t-\tau)}A_{12}^\top\dot{y}(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^t A_{12}\mathrm{e}^{A_{22}(t-\tau)}A_{12}^\top y(\tau)\,\mathrm{d}\tau + A_{11}y(\tau)\\ =& -(A_{12}\cdot(-A_{22})^{-1/2})\int_0^t \mathrm{e}^{-A_{22}(t-\tau)}(A_{12}\cdot(-A_{22})^{-1/2})^\top\dot{y}(\tau)\,\mathrm{d}\tau,\end{aligned}$$

and we obtain from Lemma A.1 that

$$\begin{aligned}&\left\|\dot{y}(\cdot) - \int_0^t A_{12}\mathrm{e}^{A_{22}(\cdot-\tau)}A_{12}^\top y(\tau)\,\mathrm{d}\tau + A_{11}y(\cdot)\right\|_{L^\infty(\mathbb{R}_{\ge 0})}\\ \le&(1+|(A_{12}\cdot(-A_{22})^{-1/2})A_{22}^{-1}(A_{12}\cdot(-A_{22})^{-1/2})^\top|)\cdot\|\dot{y}\|_{L^\infty(\mathbb{R}_{\ge 0})}\\ \le&(1+A_{12}A_{22}^{-2}A_{12}^\top)\cdot\|y\|_{W^{1,\infty}(\mathbb{R}_{\ge 0})}.\end{aligned} \tag{A.2}$$

Using the Schur complement [11, p. 103], we obtain

$$\begin{aligned}&\left(\begin{bmatrix}1 & 0_{1,n-1}\end{bmatrix}\begin{bmatrix}sI-A_{11} & -A_{12}\\ -A_{12}^\top & sI-A_{22}\end{bmatrix}^{-1}\begin{bmatrix}1\\ 0_{n-1,1}\end{bmatrix}\right)^{-1}\\ =& sI - A_{11} - A_{12}(sI-A_{22})^{-1}A_{12}^\top = sI - A_{12}(A_{22}^{-1}+(sI-A_{22})^{-1})A_{12}^\top.\end{aligned}$$

Now de l'Hôpital's rule gives rise to

$$\begin{aligned}&\lim_{s\to 0}\frac{1}{s}\left(\begin{bmatrix}1 & 0_{1,n-1}\end{bmatrix}\begin{bmatrix}sI-A_{11} & -A_{12}\\ -A_{12}^\top & sI-A_{22}\end{bmatrix}^{-1}\begin{bmatrix}1\\ 0_{n-1,1}\end{bmatrix}\right)^{-1}\\ =& 1 - A_{12}\lim_{s\to 0}\frac{1}{s}(A_{22}^{-1}+(sI-A_{22})^{-1})A_{12}^\top = 1 + A_{12}A_{22}^{-2}A_{12}^\top.\end{aligned} \tag{A.3}$$

Finally, combining (A.2) with (A.3), we obtain the desired result.

□

Appendix B. Supplements to the operator A_K.

THEOREM B.1. *Let H be a Hilbert space, which is continuously and densely embedded into the Hilbert space X and let $a : H \times H \to \mathbb{C}$ be a continuous, symmetric sesquilinear form. If, for some $\alpha > 0$, the form fulfills*

$$\operatorname{Re} a(x,x) + \langle x,x\rangle_X = a(x,x) + \langle x,x\rangle_X \geq \alpha\|x\|_X \qquad \forall x \in X, \tag{B.1}$$

then the following holds

(i) The operator

$$\begin{aligned} D(A) &:= \left\{x \in H \;\middle|\; \exists z(x) \in X : a(x,\psi) = \langle z(x),\psi\rangle_X \quad \forall \psi \in H\right\}, \\ Ax &:= -z(x) \qquad \forall x \in D(A) \end{aligned}$$

is well-defined, self-adjoint, generates an analytic semigroup in X.

(ii) $D(A)$ is dense in H with respect to $\|\cdot\|_H$.

(iii) If A is nonpositive, the operator root (in the sense of [21]) of $-A$ fulfills

$$D((-A)^{\frac{1}{2}}) = H, \qquad \langle (-A)^{\frac{1}{2}}x, (-A)^{\frac{1}{2}}y\rangle_X = a(x,y) \qquad \forall x,y \in D(A).$$

We call A the operator associated to the sesquilinear form $a(\cdot,\cdot)$.

Proof. The first part of this is [2, Thm. 4.3]. Assertions (ii) and (iii) are contained in Kato's First and Second Representation Theorem [21, Sec. VI.2]. □

LEMMA B.2. *Let $K > 0$ and define A_K as in Lemma 5.2. Then the associated bilinear form to A_K has the domain $D(a_K) = D((-A_K)^{\frac{1}{2}}) = W^{1,2}(\Omega)$ and is given by*

$$a_K(x,\psi) = \int_\Omega \nabla x(\xi)\overline{\nabla\psi(\xi)}\,\mathrm{d}\xi + K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi \cdot \int_{\partial\Omega} \overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi. \tag{B.2}$$

Proof. It is easy to see that a_K is a continuous, symmetric sesquilinear form on $H = W^{1,2}(\Omega)$ which satisfies (B.1). Hence, a_K fulfills the prerequisites of Theorem B.1 and to complete our proof it suffices to show that the domain $D(A_K)$ defined in (5.3) satisfies

$$D(A_K) = \left\{x \in W^{1,2}(\Omega) \,\middle|\, \exists z \in L^2(\Omega) : a(x,\psi) = \langle z,\psi\rangle_{L^2(\Omega)} \;\forall \psi \in W^{1,2}(\Omega)\right\}. \tag{B.3}$$

"$\subset$": Let $x \in D(A_K)$. Then Gauss' Theorem implies that for all $\psi \in W^{1,2}(\Omega)$ holds

$$\begin{aligned} a_K(x,\psi) &= \int_\Omega \nabla x(\xi)\overline{\nabla\psi(\xi)}\,\mathrm{d}\xi + K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi \cdot \int_{\partial\Omega} \overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi \\ &= -\int_\Omega \Delta x(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi + \int_{\partial\Omega} \partial_\nu x(\xi)\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi + K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi \cdot \int_{\partial\Omega} \overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi \\ &= -\int_\Omega \Delta x(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi + \left(-K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi + K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi\right)\int_{\partial\Omega} \overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi \\ &= -\int_\Omega \Delta x(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi. \end{aligned}$$

This shows the inclusion because $-\Delta x$ is an element of $L^2(\Omega)$.

"$\supset$": Let x be an element of the right hand set in (B.3). Then in particular for all infinitely often differentiable and compactly supported $\psi : \Omega \to \mathbb{C}$ the equation

$$\int_\Omega x(\xi)\overline{\Delta\psi(\xi)}\,\mathrm{d}\xi = -a_K(x,\psi) = -\int_\Omega z(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi$$

holds the by Gauss' Theorem. This implies that $\Delta x = -z \in L^2(\Omega)$. In order to show that x is in $W^{2,2}(\Omega)$, we pick some function $h \in W^{2,2}(\Omega)$ that satisfies $\partial_\nu h(\zeta) = -K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi$ for all $\zeta \in \partial\Omega$. Then for all $\psi \in W^{1,2}(\Omega)$ the following holds:

$$\begin{aligned}
&\int_\Omega \nabla(x-h)(\xi)\overline{\nabla\psi(\xi)}\,\mathrm{d}\xi\\
&= \int_\Omega \nabla x(\xi)\overline{\nabla\psi(\xi)}\,\mathrm{d}\xi - \int_\Omega \nabla h(\xi)\overline{\nabla\psi(\xi)}\,\mathrm{d}\xi\\
&= a_K(x,\psi) - K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi \cdot \int_{\partial\Omega}\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi + \int_\Omega \Delta h(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi - \int_{\partial\Omega}\partial_\nu h(\xi)\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi\\
&= a_K(x,\psi) + \int_\Omega \Delta h(\xi)\nabla\overline{\psi(\xi)}\,\mathrm{d}\xi\\
&= -\int_\Omega \Delta(x-h)(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi.
\end{aligned}$$

This implies by [13, Prop. 5.26 (ii)] that $x - h \in W^{2,2}(\Omega)$ and therefore we conclude $x \in W^{2,2}(\Omega)$. With this information we can finally apply the Gauss' Theorem which yields

$$\begin{aligned}
a_K(x,\psi) &= \int_\Omega \nabla x(\xi)\overline{\nabla\psi(\xi)}\,\mathrm{d}\xi + K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi \cdot \int_{\partial\Omega}\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi\\
&= -\int_\Omega \Delta x(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi + \int_{\partial\Omega}\partial_\nu x(\xi)\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi + K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi \cdot \int_{\partial\Omega}\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi.
\end{aligned}$$

The left hand side is by assumption equal to $-\int_\Omega \Delta x(\xi)\overline{\psi(\xi)}\,\mathrm{d}\xi$, so we have

$$\int_{\partial\Omega}\partial_\nu x(\xi)\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi + K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi \cdot \int_{\partial\Omega}\overline{\psi(\xi)}\,\mathrm{d}\sigma_\xi = 0 \quad \forall \psi \in W^{1,2}(\Omega).$$

This implies $\partial_\nu x \equiv -K\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi$. ∎

Lemma B.3. *Let $\theta \in [0,1]$ and denote by $(\cdot)'$ the duality with respect to the pivot space $L^2(\Omega)$. Then $\mathfrak{A}_K(t)$ maps $W^{\theta,2}(\Omega)'$ into $W^{1,2}(\Omega)$ and*

$$\|\mathfrak{A}_K(t)x\|_{W^{1,2}(\Omega)} \le c\left(1 + t^{-\frac{1+\theta}{2}}\right)e^{-\omega t}\|x\|_{W^{\theta,2}(\Omega)'} \qquad \forall x \in W^{\theta,2}(\Omega)' \tag{B.4}$$

with some constants $c, \omega > 0$.

Proof. We use the complex interpolation functor $[\cdot,\cdot]_\theta$ as defined in [25, Sec. 1.9.2]. With the self-adjointness of A_K it follows from [25, Sect. 1.18.10] that

$$D((-A_K)^{\theta/2}) = [L^2(\Omega), D((-A_K)^{1/2})]_\theta = [L^2(\Omega), W^{1,2}(\Omega)]_\theta.$$

According to [25, Sec. 4.3.1, Thm. 1] and [25, Eq. 2.4.2/11] the space on the right hand side of this equation is equal to $W^{\theta,2}(\Omega)$. So we have

$$D((-A_K)^{\theta/2}) = W^{\theta,2}(\Omega) \qquad \forall \theta \in [0,1].$$

Consequently, the dual spaces satisfy

$$D((-A_K)^{\theta/2})' = W^{\theta,2}(\Omega)' \qquad \forall \theta \in [0,1].$$

Then, by [24, Thm. 3.10.11], $\mathfrak{A}_K : \mathbb{R}_{\geq 0} \to \mathcal{B}(L^2(\Omega))$ extends to an analytic semigroup $\mathfrak{A}_K : \mathbb{R}_{\geq 0} \to \mathcal{B}\left(D((-A_K)^{\theta/2})'\right)$, whose generator has the domain $D((-A_K)^{1-\theta/2})$. By [24, Lem. 3.10.9], this extended semigroup maps $D(A_K^{\theta/2})'$ into $D(A_K^{1-\theta/2}) \subset D((-A_K)^{1/2})$ and there exists some $c \in \mathbb{R}_{>0}$ with

$$\|\mathfrak{A}_K(t)x\|_{D((-A_K)^{1/2})} \leq c\left(1 + t^{-\frac{1+\theta}{2}}\right) \mathrm{e}^{-\omega t} \|x\|_{D((-A_K)^{\theta/2})'} \qquad \forall x \in D((-A_K)^{\theta/2})'.$$

This proves the claim because we have shown that these spaces coincide with the corresponding Sobolev spaces. □

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