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The p-spectral radius of the Laplacian matrix

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Abstract

The p-spectral radius of a graph $G = (V, E)$ with adjacency matrix A is defined as $\lambda^{(p)}(G) = \max\{x^T Ax : \|x\|_p = 1\}$. This parameter shows remarkable connections with graph invariants, and has been used to generalize some extremal problems. In this work, we extend this approach to the Laplacian matrix L, and define the p-spectral radius of the Laplacian as $\mu^{(p)}(G) = \max\{x^T Lx : \|x\|_p = 1\}$. We show that $\mu^{(p)}(G)$ relates to invariants such as maximum degree and size of a maximum cut. We also show properties of $\mu^{(p)}(G)$ as a function of p, and a upper bound on $\max_{G:\,|V(G)|=n} \mu^{(p)}(G)$ in terms of $n = |V|$ for $p > 2$, which is attained if n is even.
Keywords: Laplacian Matrix, p-spectral radius

1 Introduction and main results

Let $G = (V, E)$ be a simple n-vertex graph with at least one edge, and with adjacency matrix A and Laplacian matrix L. We recall that $L = D - A$, where D is the diagonal matrix of vertex degrees.

It is well known that obtaining the least and the largest eigenvalues (which we denote λ_1 and λ_n, respectively) of a real symmetric matrix $M \in \mathbb{R}^{n\times n}$ can be viewed as an optimization problem using the Rayleigh-Ritz Theorem [8, Theorem 4.2.2]:

$$\lambda_1(M) = \min_{\|x\|=1} x^T Mx \leq \frac{x^T Mx}{x^T x} \leq \max_{\|x\|=1} x^T Mx = \lambda_n,$$

where $x \in \mathbb{R}^n$. Using the fact that $x^T Ax = 2\sum_{ij\in E} x_i x_j$, Keevash, Lenz and Mubayi [10] replaced the Euclidean norm $\|x\|$ by the p-norm $\|x\|_p$, where $p \in [1, \infty]$, and defined the *p-spectral radius* $\lambda^{(p)}(G)$:

$$\lambda^{(p)}(G) = \max_{\|x\|_p=1} 2\sum_{ij\in E} x_i x_j.$$

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This parameter shows remarkable connections with some graph invariants. For instance, $\lambda^{(1)}(G)$ is equal to the Lagrangian $\mathfrak{L}_G$ of G, which was defined by Motzkin and Straus [14] and satisfies $2\mathfrak{L}_G - 1 = 1/\omega(G)$, where $\omega(G)$ is the clique number of G. Obviously $\lambda^{(2)}(G)$ is the usual spectral radius, and it can be shown that $\lambda^{(\infty)}(G)/2$ is equal to the number of edges of G.

An interesting result involving this parameter is about K_r*-free graphs*, that is, graphs that do not contain a complete graph with r vertices as a subgraph. Turán [18] proved that, for all positive integers n and r, the balanced complete r-partite graph, known as a *Turán graph* $T_r(n)$, is the only graph with maximum number of edges among all K_{r+1}-free graphs of order n. Kang and Nikiforov [9] proved that, for $p \geq 1$, the graph $T_r(n)$ is also the only graph that maximizes $\lambda^{(p)}(G)$ over K_{r+1}-free graphs of order n, thus generalizing Turán's result (which is the case $p = \infty$). Other results were obtained and extended to hypergraphs [15].

This motivates us to extend this approach to the Laplacian matrix L. As $x^T Lx = \sum_{ij\in E}(x_i - x_j)^2$, we define the p-spectral radius of the Laplacian matrix as follows:

Definition 1. *Let $G = (V, E)$. The p-spectral radius of the Laplacian matrix of G is given by*

$$\mu^{(p)}(G) = \max_{\|x\|_p=1} \sum_{ij\in E} (x_i - x_j)^2.$$

According to Mohar [13], the Laplacian matrix is considered to be more natural than the adjancency matrix. It is a discrete analog of the Laplace operator, which is present in many important differential equations. The Kirchhoff Matrix-Tree theorem is an early example of the use of L in Graph Theory. The largest eigenvalue (spectral radius) of L has been associated, for example, with degree sequences of a graph [2, 7, 11, 16]. The second smallest eigenvalue and its associated eigenvectors have also been studied since the seminal work by Fiedler [6], which has been used in graph partitioning and has led to an extensive literature in spectral clustering. For more information about this area, see the survey [19] and the references therein.

Therefore we hope that the definition of $\mu^{(p)}$ will shed some light on classical parameters of graph theory. In fact, we show that, in the same fashion as $\lambda^{(p)}(G)$, the parameter $\mu^{(p)}(G)$ relates to graph invariants, such as the maximum degree and the size of a maximum cut. We also show some properties of $\mu^{(p)}(G)$ as a function of p. The main results are:

{t:main}

Theorem 1. *Let $G = (V, E)$ be a graph with at least one edge. Then*

(a) $\mu^{(1)}(G)$ is equal to the maximum degree of G;

(b) $\mu^{(\infty)}(G)/4$ is equal to the size of a maximum cut of G.

(c) The function $f_G : [1, \infty) \to \mathbb{R}$ defined by $f_G(p) = \mu^{(p)}(G)$ is strictly increasing, continuous and converges when $p \to \infty$;

It seems to be the case that, by varying p, the vector x that achieves $\mu^{(p)}(G)$ defines a maximum cut of the graph under different restrictions. For instance, $\mu^{(1)}(G)$ leads to a maximum cut with the constraint that one of the classes is a singleton, while $\mu^{(\infty)}(G)$ is gives a maximum cut with no additional constraint. A rigorous basis for this statement remains a question for further investigation.

From the computational complexity point of view, it is interesting to note that computing $\mu^{(1)}(G)$ is easy (can be done in linear time), while computing $\mu^{(\infty)}(G)$ is an NP-complete problem, it is equivalent to finding the size of a maximum cut of G. For $\lambda^{(p)}$, the opposite happens: finding $\lambda^{(1)}(G)$ is NP-complete (equivalent to finding the clique number of G), while $\lambda^{(\infty)}(G)$ can be found in linear time.

We also present an upper bound on $\mu^{(p)}(G)$ if $p \geq 2$, which is attained for even n.

{t:mu_subg_bip}

Theorem 2. *Let $G = (V, E)$ be a graph with $n = |V|$. Then for $p > 2$,*

$$\mu^{(p)}(G) \leq n^{2-2/p}.$$

If n is even, equality holds if and only if G contains $K_{n/2,n/2}$ as subgraph.

Note that this means that, for even n, the value of $\mu^{(p)}(K_n)$ is the same as the value for the balanced complete bipartite graph with n vertices. We conjecture that this holds for all n.

This paper is organized as follows. In the remainder of the section we introduce some notation. In sections 2 and 3 we prove Theorems 1 and 2, respectively. In section 4 we present some additional remarks, conjectures and questions for future research.

Before proving our results, we set the notation used throughout the paper. The objective function of our optimization problems is

$$F_G(x) = x^T L x = \sum_{ij \in E(G)} (x_i - x_j)^2.$$

We may drop the subscript of F_G if G is clear from context. It can be readily seen that $F_{G'}(x) \leq F_G(x)$ for a subgraph G' of G, and so $F_G(x) \leq F_{K_n}(x)$ for any n-vertex graph G. Furthermore, $F_G(x) = 0$ if x is constant in each connected component of G.

Finally, given an n-vertex graph $G = (V, E)$ and a vector $x \in \mathbb{R}^n$, the vertex sets P, N and Z are those on which x_i is positive, negative, or equal to zero, respectively. We write d_i for the degree of vertex i, and d_{ij} is the number of edges between vertices i and j (so $d_{ij} \in \{0, 1\}$). The all-ones vector in $\mathbb{R}^n$ is e and the i-th vector of the canonical basis of $\mathbb{R}^n$ is e_i.

2 Proof of Theorem 1

{s:t1}

In this section, we prove Theorem 1, which relates $\mu^{(p)}(G)$ to graph invariants and gives properties of $\mu^{(p)}(G)$ as a function of p. Item (a) states that $\mu^{(1)}(G)$ is equal to the maximum degree of G. In order to prove it, we need two lemmas.

{mu1a}

Lemma 2.1. *Let $x \in \mathbb{R}^n$ such that $\|x\|_1 = 1$ and $F_G(x) = \mu^{(1)}(G)$. Then at most one entry of x and of $-x$ is positive.*

Proof. Let x be as above, and assume that x or $-x$ has at least two positive coordinates. Without loss of generality, suppose $a, b \in P$ and define x' and x'' as

$$x'_k = \begin{cases} x_a + x_b & \text{if } k = a; \\ 0 & \text{if } k = b; \\ x_k & \text{otherwise.} \end{cases} \quad \text{and} \quad x''_k = \begin{cases} 0 & \text{if } k = a; \\ x_a + x_b & \text{if } k = b; \\ x_k & \text{otherwise.} \end{cases}$$

Consider the differences $\Delta' = F(x') - F(x)$ e $\Delta'' = F(x'') - F(x)$.

$$\frac{\Delta'}{x_b} = (d_a - d_{ab})(2x_a + x_b) - \sum_{aj \in E, j \neq b} 2x_j - (d_b - d_{ab})x_b + \sum_{bj \in E, j \neq a} 2x_j + 4d_{ab}x_a$$

The expression for Δ''/x_a can be readily obtained switching the roles of a and b. As $x_a, x_b > 0$ we can take

$$\frac{\Delta'}{x_b} + \frac{\Delta''}{x_a} = (d_a + d_b + 2d_{ab})(x_a + x_b) > 0,$$

so that at least one of the differences Δ' and Δ'' is positive. This contradicts the maximality of x. □

In particular, Lemma 2.1 implies that the vector $x \in \mathbb{R}^n$ that achieves $\mu^{(1)}(G)$ satisfies $\max\{|P|, |N|\} \leq 1$. Now we consider the case $|P| = |N| = 1$.

{mu1b}

Lemma 2.2. *Let $x \in \mathbb{R}^n$ such that $\|x\|_1 = 1$, $P = \{a\}$, $N = \{b\}$ and $d_a \geq d_b$. Then $d_a = F(e_a) \geq F(x)$, with equality if and only if $d_a = d_b = d_{ab}$.*

Proof. Note that $x_a^2 + x_b^2 < 1$, because $|x_a| + |x_b| = 1$. Then

$$\begin{aligned} F(x) &= d_a x_a^2 + d_b x_b^2 + d_{ab}(1 - x_a^2 - x_b^2) \\ &\leq d_a(x_a^2 + x_b^2) + d_{ab}(1 - x_a^2 - x_b^2) \leq d_a = F(e_a). \end{aligned}$$

The first and second inequalities become equalities if and only if $d_a = d_b$ and $d_a = d_{ab}$, respectively. □

Since for $x = e_a$ we have $\sum_{ij \in E}(x_i - x_j)^2 = d_a$, it follows that $\mu^{(1)}(G)$ is obtained for a vector e_a for a vertex a with maximum degree. That proves item (a) of Theorem 1. Note that the solutions are always of this form if the maximum degree is at least 2, because the equality situation of Lemma 2.2 is of interest only if the maximum degree is one. For instance, for $G = K_2$, any feasible vector attains the maximum.

Now we proceed to prove item (b), which states that $\mu^{(\infty)}(G)/4$ is equal to the size of a maximum cut of G. In this case, the problem is of the form

$$\mu^{(\infty)}(G) = \max_{\max_i |x_i| = 1} \sum_{ij \in E} (x_i - x_j)^2.$$

{muinf}

Lemma 2.3. *Let $x \in \mathbb{R}^n$ such that $\max_i |x_i| = 1$ and $F_G(x) = \mu^{(\infty)}(G)$. Then $|x_i| = 1$, for all $i \in V$.*

Proof. Let x be as stated above. Suppose that there is $a \in V$ with $-1 < x_a < 1$. Define $x', x'' \in \mathbb{R}^n$ as

$$x'_i = \begin{cases} 1 & \text{if } i = a; \\ x_i & \text{otherwise.} \end{cases} \quad \text{and} \quad x''_i = \begin{cases} -1 & \text{if } i = a; \\ x_i & \text{otherwise.} \end{cases}$$

Consider the differences $\Delta' = F(x') - F(x)$ and $\Delta'' = F(x'') - F(x)$. Then

$$\Delta' = d_a(1 - x_a^2) - 2(1 - x_a) \sum_{aj \in E} x_j$$

and similarly

$$\Delta'' = d_a(1 - x_a^2) + 2(1 + x_a) \sum_{aj \in E} x_j,$$

and therefore

$$\frac{\Delta'}{1 - x_a} + \frac{\Delta''}{1 + x_a} = 2d_a > 0.$$

So at least one of the differences Δ' and Δ'' is positive. This contradicts the maximality of x. □

Now for a vector x in the form given by Lemma 2.3 let $S = \{i \in V : x_i = 1\}$ and $T = \{i \in V : x_i = -1\}$. If $\mathrm{cut}(S, T)$ denotes the number of edges with one endpoint in S and the other in T, we have

$$F(x) = \sum_{i \in S, j \in T} (x_i - x_j)^2 = 4\mathrm{cut}(S, T).$$

Then of course $F_G(x) = \mu^{(\infty)}(G)$ if $\mathrm{cut}(S, T)$ is a maximum cut. That proves item (a) of Theorem 1. Also, the maximum value of $\mu^{(\infty)}(G)$ among graphs of order n is

$$\mu^{(\infty)}(K_n) = \mu^{(\infty)}(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) = \begin{cases} n^2 & \text{if } n \text{ is even;} \\ n^2 - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Finally we prove item (c), which shows properties of the function $f_G : [1, \infty) \to \mathbb{R}$ defined by $f_G(p) = \mu^{(p)}(G)$. Namely, the function is strictly increasing (Lemma 2.6), continuous (Lemma 2.7) and converges when $p \to \infty$ (Lemma 2.8). We denote the *p-th power mean* of $x \in \mathbb{R}^n$ as

$$M_p(x) = \left(\frac{1}{n} \sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

and recall the Power Mean Inequality [4, p. 202, Theorem 1] that states that, for $r, s \in \mathbb{R}$,

$$r > s \Longrightarrow M_r(x) \geq M_s(x),$$

with equality if and only if $|x_1| = |x_2| = \cdots = |x_n|$.

First we state two technical lemmas that will be useful.

{pnorm_bound}

Lemma 2.4. *Let $r > s \geq 1$. Then for $x \in \mathbb{R}^n$,*

$$\|x\|_r \leq \|x\|_s \leq n^{\frac{1}{s} - \frac{1}{r}} \|x\|_r.$$

Furthermore, for a nonzero vector x^* that attains the upper bound, we must have $|x_i^*| = n^{-1/r}\|x\|_r$ for all i.

Proof. Without loss of generality, we can assume that x has positive entries and $\|x\|_r = 1$. The lower bound holds because the p-norm is nonincreasing in p, and it is only achieved by e_i. The upper bound comes from the fact that, by the power mean inequality, $M_r(x) \geq M_s(x)$, or alternatively $n^{-1/r}\|x\|_r \geq n^{-1/s}\|x\|_s$, with equality if and only if all entries are equal to $n^{-1/r}$. □

{mu2p_lbound}

Lemma 2.5. *Let $G = (V, E)$ be a graph, $p > 2$ and $x \in \mathbb{R}^n$ with $\|x\|_p = 1$. Then $F_G(x) \leq n^{1-2/p}\mu^{(2)}(G)$.*

Proof. By the Rayleigh-Ritz theorem, we have $F_G(x) \leq \|x\|_2^2 \mu^{(2)}(G)$ for $x \neq 0 \in \mathbb{R}^n$. Using Lemma 2.4 with $r = p$ and $s = 2$, we obtain $\max_{x:\|x\|_p=1} \|x\|_2 = n^{1/2-1/p}$, and therefore $F_G(x) \leq n^{1-2/p}\mu^{(2)}(G)$. □

The remainder of the proof will be broken down into three lemmas, one for each claim in (c).

{mu_cresc}

Lemma 2.6. *For a graph G with at least one edge and $p \geq 1$, $\mu^{(p)}(G)$ is strictly increasing in p.*

Proof. First assume that $p' > p > 1$, and let $x \in \mathbb{R}^n$ such that $\|x\|_p = 1$ and $F(x) = \mu^{(p)}(G)$. Define $x' := x/\|x\|_{p'}$. As $\|x\|_{p'} \leq 1$, we have

$$\mu^{(p')}(G) \geq F(x') = \frac{1}{\|x\|_{p'}^2} F(x) \geq \mu^{(p)}(G). \tag{2.1}$$ {inc_quota}

As G has at least one edge ij, $\mu^{(p)}(G) > 0$; pick x such that $x_i = -x_j = 2^{-1/p}$, and $x_i = 0$ otherwise. Equality holds in equation (2.1) if and only if $x = e_i$ for some i. We argue now that for $p > 1$, e_i never attains the maximum, so that $\mu^{(p)}(G)$ is strictly increasing.

For $p > 1$, the KKT stationarity conditions of the problem are $Lx = \lambda \nabla_x(|x_1|^p + \cdots + |x_n|^p - 1)$. Note that $x \to |x|^p$ is differentiable for $p > 1$. The condition corresponding to $\partial/\partial x_j$ is

$$d_j x_j - \sum_{jk \in E} x_k = \begin{cases} p|x_j|^{p-1}\mathrm{sign}\,(x_j)\,, & \text{if } x_j \neq 0; \\ 0, & \text{if } x_j = 0. \end{cases} \tag{2.2}$$ {lagrange_j}

If i is an isolated vertex, then $F(e_i) = 0$ and optimality is not attained. Now assume that i has a neighbour j. Taking $x = e_i$, then $x_k = 0$ if $k \neq i$; in particular, $x_j = 0$. Then the right hand side of (2.2) is 0, and the left hand side is $d_j x_j - \sum_{jk \in E} x_k = 0 - x_i = -1$. Therefore, e_i does not satisfy the optimality conditions of the problem, so that, for any $i \in V$, $F(e_i) < \mu^{(p)}(G)$ for $p > 1$.

With this last statement in mind, recall that, by the proof of item (a) of Theorem 1, $\mu^{(1)}(G) = F(e_i)$ for i with maximum degree. Therefore, we conclude that $\mu^{(1)}(G) < \mu^{(p)}(G)$ for $p > 1$. This completes the proof.

□

{mu_cont}

Lemma 2.7. *For any graph G and $p \geq 1$, the function $p \to \mu^{(p)}(G)$ is continuous.*

Proof. Let $p' > p \geq 1$, and let $x' \in \mathbb{R}^n$ such that $\|x'\|_{p'} = 1$ and $F(x') = \mu^{(p')}(G)$. By Lemma 2.4, we have $\|x'\|_p \leq n^{\frac{1}{p}-\frac{1}{p'}}\|x'\|_{p'}$. Define $x := x'/\|x'\|_p$. Then

$$\mu^{(p')}(G) = F(x') = \|x'\|_p^2 F(x) \leq \; n^{\frac{2}{p}-\frac{2}{p'}}\|x'\|_{p'}\mu^{(p)}(G) = n^{\frac{2}{p}-\frac{2}{p'}}\mu^{(p)}(G)$$

By Lemma 2.6, we know that $\mu^{(p')}(G) > \mu^{(p)}(G) > 0$. It is well-known (check for example [5]) that $\mu^{(2)}(G) \leq \mu^{(2)}(K_n) = n$. Combining this with Lemma 2.5, we have $\mu^{(p)}(G) \leq n^{2-2/p}$ for $p \geq 2$; as $\mu^{(p)}(G)$ is strictly increasing in p (Lemma 2.6), this bound holds for $p \geq 1$. So

$$\begin{aligned}\mu^{(p')}(G) - \mu^{(p)}(G) &\leq n^{\frac{2}{p}-\frac{2}{p'}}\mu^{(p)}(G) - \mu^{(p)}(G) \\ &\leq \left(n^{\frac{2}{p}-\frac{2}{p'}} - 1\right) n^{2-2/p} \\ &< (n^{2(p'-p)} - 1)n^2.\end{aligned}$$

So we have $\mu^{(p')}(G) - \mu^{(p)}(G) < \epsilon$ if $p' - p < \frac{1}{2}\log_n(\epsilon/n^2 + 1)$. □

{mu_limite}

Lemma 2.8. *For any graph G,*

$$\lim_{p\to\infty} \mu^{(p)}(G) = \mu^{(\infty)}(G).$$

Proof. For a given p, let x such that $\|x\|_p = 1$ and $F(x) = \mu^{(p)}(G)$. By the proof of Lemma 2.6, we know that $x \neq e_i$, so $\max_i |x_i| < 1$. Define $x' := x/\max|x_i|$. We can choose $N = N(x') \in \mathbb{N}$ such that

$$\mu^{(p)}(G) = F(x) = (\max|x_i|)^2 F(x') > (\max|x_i|)^N \mu^{(\infty)}(G),$$

so that $0 < \mu^{(\infty)}(G) - \mu^{(p)}(G) < (1 - (\max|x_i|)^N)\mu^{(\infty)}(G)$. One can check that $\max|x_i| \geq n^{-1/p}$. We conclude the proof noting that

$$0 < \mu^{(\infty)}(G) - \mu^{(p)}(G) < (1 - n^{-N/p})\mu^{(\infty)}(G),$$

and $n^{-N/p} \to 1$ when $p \to \infty$.

□

3 Proof of Theorem 2

{s:t2}

In this section we prove Theorem 2, which establishes the upper bound $\mu^{(p)}(G) \leq n^{2-2/p}$ for $p \geq 2$, as well as a necessary and sufficient condition for equality. We denote $G = (S, T, E)$ a bipartite graph with vertex classes S and T and edge set E. First we state three auxiliary lemmas.

{mu1_same_sign}

Lemma 3.1. *Let $G = (S, T, E)$ be a bipartite graph, and $x \in \mathbb{R}^n$ such that $\|x\|_p = 1$ and $F(x) = \mu^{(p)}(G)$. Then for x or $-x$ we have $P \subseteq S$ and $N \subseteq T$.*

Proof. Let x be as stated above. Note that we can freely change the signs of the entries preserving feasibility. Without loss of generality, if we change the signs of negative entries in S and positive entries in T, we are replacing, in the sum of F, terms of the form $(|x_i| - |x_j|)^2$ by $(|x_i| + |x_j|)^2$, thus increasing F. □

{mu2_bip_same_v

Lemma 3.2. *Let $G = (S, T, E)$ be a complete bipartite graph, $p > 2$, and $x \in \mathbb{R}^n$ such that $\|x\|_p = 1$ and $F(x) = \mu^{(p)}(G)$. If i and j are in the same class of the bipartition, then $x_i = x_j$.*

Proof. By Lemma 3.1 we can assume, without loss of generality, that the entries of x corresponding to vertices in S and T are respectively nonnegative and nonpositive. Suppose that there are $i, j \in S$ with $x_i \neq x_j$. As G is complete bipartite,

$$F(x) = \sum_{k\in T}\sum_{i\in S}(x_i - x_k)^2.$$

Let M_p denote the power mean of $\{x_i : i \in S\}$. Consider the vector $x' \in \mathbb{R}^n$ such that $x'_i = M_p$ if $i \in S$ and $x'_i = x_i$ if $i \in T$. One can check that $\|x'\|_p = 1$. We claim that $F(x') > F(x)$, contradicting the maximality of x. We have

$$F(x') - F(x) \quad = \quad \sum_{k \in T} \sum_{i \in S} ({x'_i}^2 - x_i^2) + 2x_k(x'_i - x_i).$$

For any fixed $k \in T$, by the power mean inequality,

$$\sum_i {x'_i}^2 + 2\sum_i x'_i x'_k = |S|M_p^2 + 2|S|M_p x_k > |S|M_2^2 + 2|S|M_1 x_k = \sum_i x_i^2 + 2\sum_i x_i x_k.$$

□

This allows us to obtain a formula for complete bipartite graphs. {p:mu_bip_comp}

Lemma 3.3. *Let $G = (S, T, E)$ be a complete bipartite graph. For $p > 2$,*

$$\mu^{(p)}(G) = |S||T|(a+b)^2,$$

where

$$a = \left(|S| + |T| \left(\frac{|S|}{|T|} \right)^{\frac{p}{p-1}} \right)^{-1/p}, \quad b = \left(\frac{|S|}{|T|} \right)^{\frac{1}{p-1}} a.$$

Proof. By Lemma 3.2, we can assume that $x_i = a$ for $i \in S$ and $x_i = -b$ for $i \in T$. Then apply the method of Lagrange multipliers to the function $g(a, b) = |S||T|(a+b)^2$ constrained by $h(a, b) = |S|a^p + |T|b^p = 1$. □

Now we state a useful bound for the usual spectral radius. {p:mu2_bound}

Lemma 3.4. *Let G be a graph. Then $\mu^{(2)}(G) \leq n$, with equality if and only if $\bar{G}$, the complement graph of G, is disconnected.*

Proof. The proof is straightforward and may be found in [11]. We remark that $\bar{G}$ being disconnected is equivalent to G containing a complete bipartite graph as a spanning subgraph. □

In the proof of item (c) of Theorem 1, the balanced complete bipartite graph is the only bipartite graph that attains the maximum for $\mu^{(\infty)}(G)$ among graphs G of order n. We now show that the same holds for $\mu^{(p)}$ if $2 < p < \infty$ if n is even. Note that this is not the case for $p = 2$ in light of Lemma 3.4.

Proof of Theorem 2. As $\mu^{(2)}(K_n) = n$, the bound $\mu^{(p)}(G) \leq n^{2-2/p}$ is a direct consequence of Lemma 2.5. By Lemma 3.3, one can check that $\mu^{(p)}(K_{n/2,n/2}) = n^{2-2/p}$. Furthermore, if $K_{n/2,n/2} \subseteq G$, we trivially have $\mu^{(p)}(G) = n^{2-2/p}$ because $F_G(x)$ will not decrease if we add edges to G.

Now let G and $x \in \mathbb{R}^n$ such that $F_G(x) = \mu^{(p)}(G) = n^{2-2/p}$. By Lemma 2.5, this implies that $\mu^{(2)}(G) = n$. Also by Lemma 2.5, as $p > 2$, we must have $|x_i| = |x_j|$ for all $i, j \in V$. Consider the sets P and N associated with G and x, and observe that $V = P \cup N$. Since $x_i - x_j = 0$ if i and j lie in the same class, we have $\mu^{(p)}(G) = \sum_{i \in P} \sum_{j \in N} (x_i - x_j)^2$. This implies that $\{i, j\} \in E(G)$ for all $i \in P$ and $j \in N$, otherwise $F_{G \cup \{i,j\}}(x) > F_G(x) = n^{2-2/p}$, contradicting Lemma 2.5. Finally, the formula in Lemma 3.3 ensures that $|x_i| = |x_j|$ if and only if $|P| = |N|$, therefore $|P| = |N| = n/2$. □

Although we conjecture that the equality condition of Theorem 2 also holds for odd n (of course with a different bound, given by Lemma 3.3), the reasoning used in the proof does not work in this case, because then the balanced complete bipartite graph does not attain the bound given by Lemma 2.5.

4 Concluding remarks

{s:conc}

As already mentioned in the introduction, when we study the p-spectral radius of the Laplacian matrix, we seem to obtain maximum cuts under different restrictions in the graph by varying p. That motivates the following broad question for further investigation:

Question 4.1. *For $p \geq 1$, which relation possibly exists between $\mu^{(p)}(G)$ and cuts (or other parameters) of G?*

Also, we proved that computing $\mu^{(1)}(G)$ can be done in linear time, while computing $\mu^{(\infty)}(G)$ is an NP-complete problem. As finding the maximum degree of G can be trivially reduced in linear time to finding the size of a maximum cut of G, it might be the case that, by increasing p, we obtain a problem that is at least as hard. This motivates the following conjecture:

Conjecture 4.2. *Let $q > p \geq 1$. The problem of finding $\mu^{(p)}(G)$ can be reduced to the problem of finding $\mu^{(q)}(G)$ in polynomial time.*

There are other approaches that seek to generalize eigenvalues via the introduction of the p-norm. Amghibech [1] introduced a non-linear operator, which he called the p-Laplacian Δ_p, that induces a functional of the form $\langle x, \Delta_p \rangle = \sum_{ij \in E} |x_i - x_j|^p$ instead of the quadratic form of the Laplacian. This functional is unbounded for $p = \infty$ over the p-norm unit ball, and the case $p = 1$ cannot be treated directly. However, the eigenvalue formulation used allows to explore eigenvalues other than the largest and the smallest: λ is said to be a p-eigenvalue of M if there is a vector $v \in \mathbb{R}^n$ such that

$$(\Delta_p x)_i = \lambda \phi_p(v_i), \quad \phi_p(x) = |x|^{p-1} \mathrm{sign}\,(x)\,.$$

The vector v is called a p-eigenvector of M associated to λ. Using this formulation, Bühler and Hein [3] proved that the cut obtained by "thresholding" (partitioning according to entries greater than a certain constant) an eigenvector associated to the second smallest eigenvalue of Δ_p converges to the optimal Cheeger cut when $p \to 1$; in practice, the case $p = 2$ is used to obtain an approximation to this cut [17, 19].

It may be possible to adapt this method to the standard Laplacian operator, which would allow us to explore a p-norm version of the second smallest eigenvalue of L, which could potentially also lead to different cuts according to the value of p.

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