BALANCED TRUNCATION MODEL REDUCTION FOR SYMMETRIC SECOND ORDER SYSTEMS – A PASSIVITY-BASED APPROACH*

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Abstract. We introduce a model reduction approach for linear time-invariant second order systems based on positive real balanced truncation. Our method guarantees asymptotic stability and passivity of the reduced order model as well as the positive definiteness of the mass and stiffness matrices. Moreover, we receive an a priori gap metric error bound. Finally we show that our method based on positive real balanced truncation preserves the structure of overdamped second order systems.

1. Introduction. In this article, we consider linear second order systems with co-located inputs and outputs of the form

$$M\ddot{p}(t) + D\dot{p}(t) + Kp(t) = Bu(t), \quad y(t) = B^\top \dot{p}(t) \tag{1.1}$$

with symmetric $M, D, K \in \mathbb{R}^{n\times n}$, where the *mass matrix* M and the *stiffness matrix* K are positive definite, while the *damping matrix* D is positive semidefinite and $B \in \mathbb{R}^{n\times m}$. In applications, the state space dimension n typically becomes unfeasibly large for simulation, optimization, or control. This causes a demand for a good approximation of the system. Therefore, an upper bound in a certain quality measure is desirable for a model order reduction method. Such a quality measure can be some norm of the error system, such as the H^∞ norm for stable systems, or the so-called *gap metric* [11, 12] between original and reduced order system. The latter expresses the distance between the input-output trajectories of two systems. On the other hand, our model has three key attributes that a reduced model should inherit, one of these being the second order structure together with the symmetries, meaning the reduced system should be of the form

$$\widetilde{M}\ddot{\widetilde{p}}(t) + \widetilde{D}\dot{\widetilde{p}}(t) + \widetilde{K}\widetilde{p}(t) = \widetilde{B}u(t), \quad \widetilde{y}(t) = \widetilde{B}^\top \dot{\widetilde{p}}(t), \tag{1.2}$$

with symmetric $\widetilde{M}$, $\widetilde{D}$, $\widetilde{K} \in \mathbb{R}^{r\times r}$, and $\widetilde{B} \in \mathbb{R}^{r\times m}$, where $r \ll n$. Other physically meaningful properties are asymptotic stability and passivity, which should also be preserved. Passivity describes the property that the energy in the system is solely induced by input and output and may be dissipated or conserved in the system.

Some progress has been made in [5, 30] and preservation of physical properties in the reduced order model is possible for co-located inputs and outputs. Further second order reduction methods have been developed in [3, 5, 7, 10, 19, 26, 32] (see also [30] for an overview). Besides these, there exists interpolatory methods, which succeed either in preserving the second order structure [33] or deliver a posteriori H^∞ error bounds [27]. However, all the approaches mentioned lack a combination of the two.

*This work has been supported by DFG priority program 1897: "Calm, Smooth and Smart – Novel Approaches for Influencing Vibrations by Means of Deliberately Introduced Dissipation".

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These approaches have all in common that the ansatz for the reduced order model is $p(t) = W_r\widetilde{p}(t)$ for some "tall matrix" $W_r \in \mathbb{R}^{n\times r}$ and an accordant multiplication of the state equation in (1.2) from the left with some "flat" matrix $V_r \in \mathbb{R}^{r\times n}$. Our approach is somewhat different from these. Namely, we first perform a reduction of the first order representation, and accordingly carry out a transformation yielding a second order model. This corresponds to a reduction ansatz $p(t) = W_{r,1}\widetilde{p}(t) + W_{r,2}\dot{\widetilde{p}}(t)$ together with a linear combination of the state equation and its derivative such that again a second order system is obtained. The basis for our considerations is the technique of *positive real balanced truncation*, which is a passivity-preserving method for first order systems, see [8]. Moreover, an a priori error bound in the gap metric is provided [17]. We present a modification of positive real balanced truncation, which preserves the second order structure, symmetry of the system matrices, stability, passivity as well as positive definiteness of the mass, and stiffness matrices. Namely, we show that, under an extra condition on the so called *zero sign characteristics*, the reduced order model obtained by positive real balanced truncation of a first order realization of (1.1) can be transformed to a second order realization of the form (1.2). To this end, we use the theory of *standard triples* by Gohberg, Lancaster and Rodman [14] to derive a second order realization (1.2) in which the mass and stiffness matrices are positive definite, and the damping matrix has at least $r - m$ positive eigenvalues provided that $r \geq m$. Furthermore, if the original system is *overdamped*, see (7.1), our method will further produce a reduced order model with positive definite damping matrix.

Our method in a nutshell. The model reduction technique presented in this article is – in theory – consisting of three steps:

Step 1: For $H, G \in \mathbb{R}^{n\times n}$ with $M = HH^\top$ $K = GG^\top$, we set $x(t) := \begin{bmatrix} G^\top H^{-\top} p(t) \\ \dot{p}(t) \end{bmatrix}$ and rewrite the second order system (1.1) as

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & G^\top H^{-\top} \\ -H^{-1}G & -H^{-1}DH^{-\top} \end{bmatrix}}_{=:\mathcal{A}} x(t) + \underbrace{\begin{bmatrix} 0 \\ B \end{bmatrix}}_{=:\mathcal{B}} u(t), \quad y(t) = \underbrace{\begin{bmatrix} 0 & B^\top \end{bmatrix}}_{=:\mathcal{C}} x(t). \tag{1.3}$$

The most important feature of this system is that it has an internal symmetry structure $\mathcal{A}\mathscr{S}_n = \mathscr{S}_n\mathcal{A}^\top$ and $\mathcal{C} = \mathcal{B}^\top = \mathcal{B}^\top\mathscr{S}_n$, where $\mathscr{S}_n := \operatorname{diag}(-I_n, I_n)$. In particular, its *transfer function* $\mathbf{G}(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B}$ is symmetric, i. e., it fulfills $\mathbf{G}(s)^\top = \mathbf{G}(s)$. We will make heavy use of this symmetry structure.

Step 2: We apply positive real balanced truncation [17] to the first order system (1.3). The internal symmetry structure of (1.3) yields that positive real balanced truncation can be done by determining only one (instead of two) solutions of the Kálmán-Yakubovich-Popov inequality. We show that the positive real characteristic values, see Definition 2.2 b), can – in a certain sense – be allocated to the symmetry structure of the system (1.3). This is the basis for our finding that the resulting first order model is – without putting any further computational effort – of the form

$$\begin{aligned} \dot{\widetilde{x}}(t) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \mathcal{A}_{16} \\ 0 & 0 & 0 & 0 & \mathcal{A}_{25} & \mathcal{A}_{26} \\ 0 & 0 & \mathcal{A}_{33} & \mathcal{A}_{34} & 0 & \mathcal{A}_{36} \\ 0 & 0 & -\mathcal{A}_{34}^\top & \mathcal{A}_{44} & 0 & \mathcal{A}_{46} \\ 0 & -\mathcal{A}_{25}^\top & 0 & 0 & 0 & 0 \\ -\mathcal{A}_{16}^\top & -\mathcal{A}_{26}^\top & -\mathcal{A}_{36}^\top & \mathcal{A}_{46}^\top & 0 & \mathcal{A}_{66} \end{bmatrix} \widetilde{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{B}_6 \end{bmatrix} u(t), \\ \widetilde{y}(t) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \mathcal{B}_6^\top \end{bmatrix} \widetilde{x}(t), \end{aligned} \tag{1.4}$$

where the block sizes from left to right and from top to bottom are m, ℓ, p, p, ℓ, m, with $r = p + m + \ell$. Note that, if $\mathcal{A}_{33}$ is zero, then it would – by merging some of the blocks – be of the form

$$\dot{\widetilde{x}}(t) = \begin{bmatrix} 0 & \widetilde{G}^\top \\ -\widetilde{G} & -\widetilde{D} \end{bmatrix} \widetilde{x}(t) + \begin{bmatrix} 0 \\ \widetilde{\mathcal{B}} \end{bmatrix} u(t), \qquad (1.5)$$
$$\widetilde{y}(t) = \begin{bmatrix} 0 & \widetilde{\mathcal{B}}^\top \end{bmatrix} \widetilde{x}(t),$$

which would result in a reduced second order model (1.2) with $\widetilde{M} = I_r$, $\widetilde{K} = \widetilde{G}\widetilde{G}^\top$ and $\widetilde{D} = \widetilde{D}^\top$. This is regrettably not the case in general, whence we apply

Step 3: We apply a state space transformation to (1.4) such that the matrix $\mathcal{A}_{33}$ vanishes. More precisely, we first intend to find some $\widehat{T} \in \mathrm{Gl}_{2p}(\mathbb{R})$ that preserves the symmetry structure, i. e., it fulfills $\widehat{T}^\top \mathscr{S}_p \widehat{T} = \mathscr{S}_p$, and

$$\widehat{T}^{-1} \begin{bmatrix} \mathcal{A}_{33} & \mathcal{A}_{34} \\ -\mathcal{A}_{34}^\top & \mathcal{A}_{44} \end{bmatrix} \widehat{T} = \begin{bmatrix} 0 & \widehat{\mathcal{A}}_{34} \\ -\widehat{\mathcal{A}}_{34}^\top & \widehat{\mathcal{A}}_{44} \end{bmatrix}. \qquad (1.6)$$

Then a state space transformation with $T = \mathrm{diag}(I_{m+\ell}, \widehat{T}, I_{\ell+m})$ leads to a system which is indeed of the form (1.5) and can then be rewritten as a second order system. To derive such a transformation, we have to use techniques from indefinite linear algebra. In particular, the aforementioned symmetry structure of our system provides that each real (invariant) zero of the system can be assigned a signature, see Definition 4.6. If the reduced system is minimal, the zeros with positive signature are given by some $\mu_1^+ \leq \ldots \leq \mu_k^+ < \mu_{k+1}^+ = \ldots = \mu_{k+m}^+ = 0$ and those with negative signature by $\mu_1^- \leq \ldots \leq \mu_k^- < 0$. Our main results on the existence and construction of such transformations are the following (see Theorem 6.7): suppose the reduced system is minimal, then the following are equivalent:

a) The system (1.4) can be written in second order form (1.5).
b) It exists a $\widehat{T} \in \mathrm{Gl}_{2p}(\mathbb{R})$ with $\widehat{T}^\top \mathscr{S}_p \widehat{T} = \mathscr{S}_p$, that fulfills (1.6).
c) For $i = 1, \ldots, k$ it holds that $\mu_i^- < \mu_i^+$.

Here, the implication "a)⇒c)" is based on [24, Thm. 16]. If the reduced system is not minimal or c) is not fulfilled, we add equations to the system in a fashion that the newly formed system is also stable, passive and such that the transfer function is preserved. Here, in the worst case the number of states, i.e., the size of the mass matrix, doubles.

Outline of the article. The article is structured as follows. In Section 2 we introduce the passivity preserving balanced truncation and some background material from systems theory. In Section 3 we present our positive real balanced truncation ansatz for second order systems and prove some of its main features. What is left, the derivation of the balanced form of second order systems, namely (1.4), is done in Section 5. Before we focus on the reconstruction of the second order structure and the proof of the necessary condition on the zeros in Section 6, Section 4 introduces some necessary concepts from indefinite linear algebra. Section 7 considers the case that the original system is overdamped. Last but not least, in Section 8 we summarize our method in a numerical procedure and hereby also discuss the numerical treatment.

Notations. The set of natural numbers including zero is denoted by $\mathbb{N}$. The symbols $\mathbb{R}[s]$ and $\mathbb{R}(s)$ respectively stand for the ring of real polynomials and the field of real rational functions. We denote by $\mathbb{C}^+$ and $\overline{\mathbb{C}^+}$ the open and closed complex half-plane. Further, $\mathbf{G}(s) \in \mathbb{R}(s)^{m\times m}$ is called *proper* if $\lim_{s\to\infty} \mathbf{G}(s) < \infty$ and *strictly* *proper* if the latter limit is zero. The transpose and conjugate transpose of $T \in \mathbb{C}^{m\times n}$ are denoted by $T^\top$ and T^*, respectively. For symmetric matrices $X, Y \in \mathbb{R}^{n\times n}$, we write $X > Y$ if $X - Y$ is positive definite and $X \geq Y$ if $X - Y$ is positive semidefinite.

We call $S = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$, where $\varepsilon_i = \pm 1$ for $i = 1, \ldots, n$, a *signature matrix* and make frequent use of the signature matrix $\mathscr{S}_n := \operatorname{diag}(-I_n, I_n) \in \mathbb{R}^{2n\times 2n}$.

The symbol $\mathrm{Gl}_n(\mathbb{R})$ stands for the set of invertible $n \times n$ matrices with entries in $\mathbb{R}$. By $\mathcal{R}H^\infty(\mathbb{C}^{m\times m})$, we denote the space of proper elements of $\mathbb{R}(s)^{m\times m}$ with entries having no poles in $\overline{\mathbb{C}^+}$, where $\mathcal{R}H^2(\mathbb{C}^m)$ the space of strictly proper elements of $\mathbb{R}(s)^{m\times m}$ with entries having no poles in $\overline{\mathbb{C}^+}$. Note that $\mathcal{R}H^\infty(\mathbb{C}^{m\times m})$ becomes a normed space when equipped with the norm

$$\|\mathbf{G}(s)\|_\infty := \sup_{\omega\in\mathbb{R}} \|\mathbf{G}(\mathrm{i}\omega)\|_2.$$

Moreover, $\mathcal{R}H^2(\mathbb{C}^m)$ is an inner product space provided with the norm

$$\|u(s)\|_2^2 := \int_{\mathbb{R}} \|u(\mathrm{i}\omega)\|_2^2 \mathrm{d}\omega.$$

Here, $\|\cdot\|_2$ on the right-hand side stands for the maximum singular value of matrices and the Euclidean norm of vectors, respectively.

2. Positive real balanced truncation of first order systems. We revisit positive real balanced truncation for linear time-invariant first order systems

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad y(t) = \mathcal{C}x(t) + \mathcal{D}u(t), \tag{2.1}$$

with $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B}, \mathcal{C}^\top \in \mathbb{R}^{n\times m}$ and $\mathcal{D} \in \mathbb{R}^{m\times m}$. The dynamical system (2.1) is said to be *minimal* if it is both controllable and observable. The *transfer function* is given by $\mathbf{G}(s) = \mathcal{C}(sI_n - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D} \in \mathbb{R}(s)^{m\times m}$. We also speak of (2.1) as a realization of $\mathbf{G}(s)$ and use the notation $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ to refer to this system, or if $\mathcal{D} = 0$ we write $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$. We call $\mu \in \mathbb{C}$ an *(invariant) zero of* $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$, if

$$\operatorname{rank}_{\mathbb{C}} \begin{bmatrix} -\mu I_n + \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} < \operatorname{rank}_{\mathbb{R}(s)} \begin{bmatrix} -sI_n + \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}.$$

In other words, the set of zeros of $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ equals to the set of eigenvalues of the matrix pencil $\left[\begin{smallmatrix} -sI_n + \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$. If the latter pencil is square and invertible as a matrix over $\mathbb{R}(s)$ (which is, by taking the Schur complement, equivalent to the transfer function being invertible), then $\mu \in \mathbb{C}$ is a zero of $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$, if and only if there exists some $v \in \mathbb{C}^{n+m} \setminus \{0\}$ with $\left[\begin{smallmatrix} -\mu I_n + \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] v = 0$.

In the following we deal with systems having *positive real* transfer functions. That is,

a) $\mathbf{G}(s)$ has no poles in $\mathbb{C}^+$.
b) $\mathbf{G}(\lambda) + \mathbf{G}(\lambda)^* \geq 0$ for all $\lambda \in \mathbb{C}^+$.

If the inequality in (ii) is strict, $\mathbf{G}(s)$ is called *strictly positive real.* If $\mathbf{G}(s)$ is positive real and invertible, also $\mathbf{G}^{-1}(s)$ is positive real. As a consequence, realizations $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ of positive real functions with the property that all eigenvalues of $\mathcal{A}$ have nonpositive real part do not have any zeros in $\mathbb{C}^+$. In particular, minimal realizations of positive real transfer functions do not have any zeros in $\mathbb{C}^+$. The famous positive real lemma draws a link between positive realness of the transfer function and the solvability of a certain linear matrix inequality.

THEOREM 2.1 (Positive real lemma [2, Chap. V]). *Let* $[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]$ *be a minimal system of the form* (2.1)*, with transfer function* $\mathbf{G}(s) \in \mathbb{R}(s)^{m\times m}$. *Then* $\mathbf{G}(s)$ *is positive real, if and only if there exists some* $P > 0$*, such that the* Kálmán-Yakubovich-Popov inequality (KYP)

$$\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(P) := \begin{bmatrix} \mathcal{A}^\top P + P\mathcal{A} & P\mathcal{B} - \mathcal{C}^\top \\ \mathcal{B}^\top P - \mathcal{C} & -\mathcal{D} - \mathcal{D}^\top \end{bmatrix} \le 0 \tag{2.2}$$

is fulfilled. Moreover, there exists a minimal *solution* $P_{\min} > 0$ *and a* maximal *solution* $P_{\max} > 0$ *of* $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(P) \le 0$*, i. e., for all other solutions* P *of* (2.2) *it holds that* $P_{\max} \ge P \ge P_{\min}$.

The KYP inequality admits the so-called *dissipation inequality*. That is, for all locally square integrable solutions (x, u, y) of $[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]$ and $t > 0$ it holds that

$$x(t)^\top P x(t) \le x(0)^\top P x(0) + \int_0^t y(\tau)^\top u(\tau)\, \mathrm{d}\tau\,. \tag{2.3}$$

Such systems are also called *passive*. Since $\mathbf{G}(s)^\top$ is the transfer function of $[\mathcal{A}^\top,\mathcal{C}^\top,\mathcal{B}^\top,\mathcal{D}^\top]$, this system is passive as well. Therefore, the dual KYP inequality $\mathscr{W}_{[\mathcal{A}^\top,\mathcal{C}^\top,\mathcal{B}^\top,\mathcal{D}^\top]}(Q) \le 0$ has again a minimal solution. Moreover, $P > 0$ solves $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(P) \le 0$, if and only if P^{-1} is a solution of $\mathscr{W}_{[\mathcal{A}^\top,\mathcal{C}^\top,\mathcal{B}^\top,\mathcal{D}^\top]}(Q) \le 0$. As a consequence, if $P_{\min}$ is the minimal such solution in the sense of Theorem 2.1, then $P_{\min}^{-1}$ is the maximal solution of the dual KYP inequality.

DEFINITION 2.2 (positive real balanced, internally passive). *With the notation of Theorem 2.1, a system* $[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]$ *is called*

a) positive real balanced*, if for the minimal solutions* $P_{\min}$*,* $Q_{\min}$ *of the KYP inequalities* $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(P) \le 0$ *and* $\mathscr{W}_{[\mathcal{A}^\top,\mathcal{C}^\top,\mathcal{B}^\top,\mathcal{D}^\top]}(Q) \le 0$ *we have*

$$P_{\min} = Q_{\min} = \operatorname{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_h I_{n_h}),$$

where $\sigma_1, \ldots, \sigma_h$ *are distinct values in* $(0, 1]$. *The latter are called the* positive real characteristic values *of* $[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]$.

b) internally passive*, if* $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(I_n) \le 0$.

In contrast to the conventional definition of positive real balanced, we do not assume that the positive real characteristic values of $[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]$ are ordered.

Note that internal passivity implies that in the dissipation inequality (2.3), the quadratic form with P is the square of the norm of the state. It has been shown in Theorem 7 of [31] that any positive real balanced realization is internally passive.

If there exist minimal solutions $P_{\min} \ge 0$, $Q_{\min} \ge 0$ of the KYP inequalities $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(P) \le 0$ and $\mathscr{W}_{[\mathcal{A}^\top,\mathcal{C}^\top,\mathcal{B}^\top,\mathcal{D}^\top]}(Q) \le 0$, then a certain state space transformation leads to a positive real balanced realization. We refer to this as *positive real balancing*. Our main emphasis is on *positive real balanced truncation*, that is balancing the system and accordingly removing the blocks corresponding to some positive real characteristic values. This leads to a reduced system which is passive and if the transfer function of the original system is strictly positive real, is again asymptotically stable and positive real balanced, see Theorem 4 or Lemma 2 and Theorem 1 in [18]. Both steps can be done at once while also removing the uncontrollable and unobservable parts [34]. Namely, by using factorizations $P_{\min} = L^\top L$, $Q_{\min} = R^\top R$ and the singular value decomposition

$$LR^\top = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \tag{2.4}$$

we are able to define the reduction matrices $W^\top = \Sigma_1^{-1/2} Z_1 L$ and $V = R^\top U_1 \Sigma_1^{-1/2}$. A reduced model received from positive real balanced truncation is given by the system $[W^\top \mathcal{A} V, W^\top \mathcal{B}, \mathcal{C} V, \mathcal{D}]$. If we do not truncate any singular values, then we obtain a minimal system as the following lemma shows.

LEMMA 2.3. *Let a system $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ with positive real transfer function $\mathbf{G}(s)$ be given. Assume that $P_{\min}$, $Q_{\min}$ are minimal solutions of the KYP inequalities $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(P) \leq 0$ and $\mathscr{W}_{[\mathcal{A}^\top,\mathcal{C}^\top,\mathcal{B}^\top,\mathcal{D}^\top]}(Q) \leq 0$, and let L and R be matrices with full row rank and $P_{\min} = L^\top L$, $Q_{\min} = R^\top R$. Further, let $LR^\top = U\Sigma Z$ be a singular value decomposition and let $W^\top = \Sigma^{-1/2} Z L$, $V = R^\top U \Sigma^{-1/2}$. Then $[W^\top \mathcal{A} V, W^\top \mathcal{B}, \mathcal{C} V, \mathcal{D}]$ is a minimal and positive real balanced realization of $\mathbf{G}(s)$. Moreover, for any further minimal positive real balanced realization $[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}, \widehat{\mathcal{C}}, D]$ of $\mathbf{G}(s)$ with positive real characteristic values in the same order, there exist orthogonal matrices $Q_i \in \mathbb{R}^{n_i \times n_i}$ for $i = 1, \ldots, h$ such that for $T := \operatorname{diag}(Q_1, Q_2, \ldots, Q_h)$, $\widehat{\mathcal{A}} = T^{-1} \mathcal{A} T$, $\widehat{B} = T^{-1} \mathcal{B}$ and $\widehat{\mathcal{C}} = \mathcal{C} T$.*

Proof. By using an appropriate state space transformation, we can assume that $P_{\min} =: \operatorname{diag}(P_1, 0)$ for some positive definite matrix P_1, and partition $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ accordingly, i. e., $\mathcal{A} = \left[\begin{smallmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{smallmatrix}\right]$, and $\mathcal{C} = \left[\begin{smallmatrix} \mathcal{C}_1 & \mathcal{C}_2 \end{smallmatrix}\right]$. Since $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]} \leq 0$, the block form yields $\mathcal{A}_{12} = 0$ and $\mathcal{C}_2 = 0$, i. e., we obtain a Kálmán observability decomposition in which the kernel of P corresponds to the unobservable states. On the other hand, if $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ is in Kálmán observability decomposition, and $P_{\min} = \left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{smallmatrix}\right]$ according to the block structure of the Kálmán controllability decomposition, then a simple calculation shows that $P = \left[\begin{smallmatrix} P_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(P) \leq 0$. The minimality of $P_{\min}$ therefore leads to $P_{12} = 0$ and $P_{22} = 0$. As a consequence, the kernel of P indeed corresponds to the space of unobservable states. Likewise, the image of $Q_{\min}$ corresponds to the space of controllable states, whence, by using the results from [34], $[W^\top \mathcal{A} Z, W^\top \mathcal{B}, \mathcal{C} Z, \mathcal{D}]$ is a minimal positive real balanced realization of $\mathbf{G}(s)$.

The second statement is Lemma 6 from [31]. ∎

A consequence of this lemma is that the reduced transfer function does not depend on the specific minimal positive real balanced realization of the original transfer function.

Now we present details on the error bound of positive real balanced truncation. A *(right) coprime factorization* of $\mathbf{G}(s) \in \mathbb{R}(s)^{p \times m}$ is $\left[\begin{smallmatrix} \mathbf{M}(s) \\ \mathbf{N}(s) \end{smallmatrix}\right]$ consisting of $\mathbf{N}(s) \in \mathcal{R}H^\infty(\mathbb{C}^{p \times m})$, $\mathbf{M}(s) \in \mathcal{R}H^\infty(\mathbb{C}^{m \times m})$ such that $\mathbf{G}(s) = \mathbf{N}(s)\mathbf{M}(s)^{-1}$, and if there exist $\mathbf{X}(s) \in \mathcal{R}H^\infty(\mathbb{C}^{m \times m})$ and $\mathbf{Y}(s) \in \mathcal{R}H^\infty(\mathbb{C}^{m \times p})$ that satisfy the *Bézout identity* $\mathbf{X}(s)\mathbf{M}(s) + \mathbf{Y}(s)\mathbf{N}(s) = I_m$. A coprime factorization $\left[\begin{smallmatrix} \mathbf{M}(s) \\ \mathbf{N}(s) \end{smallmatrix}\right]$ is called *normalized* if additionally, $\mathbf{M}^\top(-s)\mathbf{M}(s) + \mathbf{N}^\top(-s)\mathbf{N}(s) = I_m$. Such factorizations can be computed using techniques as in [25, 37]. Considering normalized coprime factorizations, a distance measure for general transfer functions can be introduced.

DEFINITION 2.4. *Let $\mathbf{G}_1(s), \mathbf{G}_2(s) \in \mathbb{R}(s)^{p \times m}$ be given with respective normalized coprime factorizations $\left[\begin{smallmatrix} \mathbf{M}_1(s) \\ \mathbf{N}_1(s) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \mathbf{M}_2(s) \\ \mathbf{N}_2(s) \end{smallmatrix}\right]$. Let $\Pi_1, \Pi_2 : \mathcal{R}H^2(\mathbb{C}^{m+p}) \to \mathcal{R}H^2(\mathbb{C}^{m+p})$ be orthogonal projectors with*

$$\operatorname{im} \Pi_1 = \begin{bmatrix} \mathbf{M}_1(s) \\ \mathbf{N}_1(s) \end{bmatrix} \cdot H^2(\mathbb{C}^m), \qquad \operatorname{im} \Pi_2 = \begin{bmatrix} \mathbf{M}_2(s) \\ \mathbf{N}_2(s) \end{bmatrix} \cdot H^2(\mathbb{C}^m).$$

Then the gap between $\mathbf{G}_1(s)$ and $\mathbf{G}_2(s)$ *is defined via*

$$\delta_g(\mathbf{G}_1(s), \mathbf{G}_2(s)) := \|\Pi_1 - \Pi_2\|_{L(\mathcal{R}H^2(\mathbb{C}^{m+p}))},$$

where $\|\cdot\|_{L(\mathcal{R}H^2(\mathbb{C}^{m+p}))}$ *denotes the operator norm on* $\mathcal{R}H^2(\mathbb{C}^{m+p})$*.*

It is shown in [36] that $\delta_g(\cdot,\cdot)$ fulfills the axioms of a metric. The gap metric between two systems is simply the gap metric between their transfer functions. The properness of $\mathbf{M}^{-1}(s)$ ensures that the gap metric expresses a measure for the distance between the input-output trajectories of two systems. Further note that the gap metric is also applicable to unstable systems.

Positive real balanced truncation has the following gap metric error bound.

THEOREM 2.5. *[17, Cor. 2.2] Let* $[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]$ *be a realization of the positive real function* $\mathbf{G}(s)\in\mathbb{R}(s)^{m\times m}$*. Denote the positive real characteristic values by* $(\sigma_i)_{i=1}^h$ *and, for* $r<h$*, let* $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}},\mathcal{D}]$ *be obtained by positive real balanced truncation of* $[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]$ *by removing the blocks corresponding to* $\sigma_{r+1},\ldots,\sigma_h$*. Then the transfer function* $\widetilde{\mathbf{G}}(s)$ *of* $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}},\mathcal{D}]$ *fulfills*

$$\delta_g(\mathbf{G}(s),\widetilde{\mathbf{G}}(s))\le 2\sum_{i=r+1}^{h}\sigma_i.$$

Note that [17] considers positive real balanced truncation in which the states corresponding to the smallest characteristic values are removed. A careful inspection of the proof of the error bound (and those of the therein used results) yields that the above error bound still holds when states corresponding to arbitrary positive real characteristic values are removed. In our method for second order systems we will make use of this fact.

3. Positive real balanced truncation for second order systems. We introduce positive real balanced truncation for systems having a symmetric second order structure. Now we consider linear time-invariant first order systems $[\mathcal{A},\mathcal{B},\mathcal{C}]$ with $\mathcal{A}\in\mathbb{R}^{2n\times 2n}$ and $\mathcal{B}\in\mathbb{R}^{2n\times m}$ structured as in (1.3), that is

$$\mathcal{A}=\begin{bmatrix}0 & G^\top\\ -G & -D\end{bmatrix}\in\mathrm{Gl}_{2n}(\mathbb{R}),\quad \mathcal{B}=\begin{bmatrix}0\\ B\end{bmatrix}=\mathcal{C}^\top \tag{3.1}$$

for some $D\in\mathbb{R}^{n\times n}$ with $D=D^\top\ge 0$ and $B\in\mathbb{R}^{n\times m}$. The transfer function is given by

$$\mathbf{G}(s)=\mathcal{C}(sI_{2n}-\mathcal{A})^{-1}\mathcal{B}=sB^\top(s^2I_n+sD+K)^{-1}B\in\mathbb{R}(s)^{m\times m},$$

where $K=GG^\top$. We first notice the following:

Remark 3.1 (Second order systems, passivity, and zeros).

a) *We assume throughout the remaining sections that* $\operatorname{rank}\mathcal{B}=m$*. This is no restriction, since otherwise, there exists an orthogonal matrix* $T\in\mathbb{R}^{m\times m}$ *such that* $\mathcal{B}T=\begin{bmatrix}\mathcal{B}_1 & 0\end{bmatrix}$*, where* $\operatorname{rank}\mathcal{B}_1=m$*. Hence* $T^\top\mathbf{G}(s)T=\left[\begin{smallmatrix}\mathbf{G}_1(s) & 0\\ 0 & 0\end{smallmatrix}\right]$ *for some* $\mathbf{G}_1(s)\in\mathbb{R}(s)^{k\times k}$ *with* $k=\operatorname{rank}\mathcal{B}_1$*. In this case one can approximate* $\mathbf{G}_1(s)$ *instead and afterwards add the zero rows and columns to the reduced transfer function.*

b) *It can be seen that* $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{B}^\top,0]}(I_{2n})\le 0$ *and hence,* $\mathbf{G}(s)$ *is positive real by the positive real lemma (see Theorem 2.1). Then [29, Thm. 15] guarantees the existence of respective minimal solutions* $P_{\min},Q_{\min}\ge 0$ *of* $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{B}^\top,0]}(P)\le 0$ *and* $\mathscr{W}_{[\mathcal{A}^\top,\mathcal{B},\mathcal{B}^\top,0]}(Q)\le 0$*, if* $(\mathcal{A},\mathcal{B})$ *and* $(\mathcal{A}^\top,\mathcal{C}^\top)$ *are stabilizable. By using the symmetry structure of the system (3.1), i. e.,* $\mathcal{A}^\top=\mathscr{S}_n\mathcal{A}\mathscr{S}_n$ *and* $\mathcal{B}^\top=\mathcal{C}\mathscr{S}_n$ *for* $\mathscr{S}_n=\operatorname{diag}(-I_n,I_n)$*, the latter two properties are however equivalent due to*

$(\mathcal{A}^\top,\mathcal{C}^\top) = (\mathscr{S}_n\mathcal{A}\mathscr{S}_n, \mathscr{S}_n\mathcal{B})$. *Since further,* $\mathcal{A}$ *does not have any eigenvalues in* $\mathbb{C}^+$*, the absence of uncontrollable purely imaginary eigenvalues is sufficient for the existence of minimal solutions* $P_{\min}, Q_{\min} \geq 0$.

c) *The assumption that* $\operatorname{rank} B = m$ *furthermore implies that the transfer function of* $[\mathcal{A},\mathcal{B},\mathcal{C}]$ *with matrices in* (3.1) *is strictly positive real. Consequently, the transfer function* $\mathbf{G}(s) \in \mathbb{R}(s)^{m\times m}$ *is invertible. Further note that positive realness of* $\mathbf{G}(s)$ *together with* $\mathcal{A}$ *having no eigenvalues in* $\mathbb{C}^+$ *implies that the system* $[\mathcal{A},\mathcal{B},\mathcal{C}]$ *has no zeros in* $\mathbb{C}^+$.

The symmetry structure further implies that $P \geq 0$ solves $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},0]}(P) \leq 0$, if and only if $Q := \mathscr{S}_n P \mathscr{S}_n$ solves $\mathscr{W}_{[\mathcal{A}^\top,\mathcal{C}^\top,\mathcal{B}^\top,0]}(Q) \leq 0$. In particular, $Q_{\min} = \mathscr{S}_n P_{\min} \mathscr{S}_n$ and thus, for $L^\top L = P_{\min}$, we obtain that $\mathscr{S}_n L^\top L \mathscr{S}_n = Q_{\min}$. Altogether, instead of the singular value decomposition (2.4) we can compute the eigendecomposition

$$L\mathscr{S}_n L^\top = \underbrace{\begin{bmatrix} U^- & U^+ \end{bmatrix}}_{=:U} \mathscr{S}_n \underbrace{\begin{bmatrix} \Sigma^- & 0 \\ 0 & \Sigma^+ \end{bmatrix}}_{=:\Sigma} \begin{bmatrix} U^- & U^+ \end{bmatrix}^\top, \tag{3.2}$$

where $U \in \mathbb{R}^{n\times n}$ is orthogonal and

$$\begin{aligned} \Sigma^- &= \operatorname{diag}\left(\sigma_1^- I_{n_1^-}, \ldots, \sigma_{h^-}^- I_{n_{h^-}^-}\right), \qquad & 0 \leq \sigma_{h^-}^- < \ldots < \sigma_1^- \leq 1, \\ \Sigma^+ &= \operatorname{diag}\left(\sigma_{h^+}^+ I_{n_{h^+}^+}, \ldots, \sigma_1^+ I_{n_1^+}\right), \qquad & 0 \leq \sigma_{h^+}^+ < \ldots < \sigma_1^+ \leq 1. \end{aligned} \tag{3.3}$$

Next we choose some positive real characteristic values of positive and negative type which correspond to truncated states. To this end, let $r^+, r^- \in \mathbb{N}$ be such that for some $q^+, q^- \in \mathbb{N}$, $r^\pm = \sum_{j=1}^{q^\pm} n_j^\pm$. Additionally, these numbers have to be chosen such that the states corresponding to zero characteristic values are truncated, and the set of characteristic values corresponding to the truncated states does not intersect with the set of characteristic values corresponding to the preserved states. This means that

$$\begin{gathered} 1 \leq q^\pm \leq h^\pm, \qquad \sigma_{q^\pm} > 0, \\ \sigma_{q^-+j^-}^- \neq \sigma_{i^+}^+ \text{ and } \sigma_{q^++j^+}^+ \neq \sigma_{i^-}^- \text{ for all } i^\pm = 1,\ldots,q^\pm,\ j^\pm = 1,\ldots,h^\pm - q^\pm. \end{gathered} \tag{3.4}$$

Note that the above condition is only of pathological nature and does not impose any serious restriction from a numerical point of view, since generically, it holds that the characteristic values in $(0,1)$ are simple.

The general purpose is that the reduced system can be transformed into a second order system. To this end we require that the reduced system has a symmetry structure with respect to a matrix $\mathscr{S}_r$. Hence it is essential that we find $r^\pm$ which additionally fulfill $r^- = r^+$. We partition (3.2) as

$$L\mathscr{S}_n L^\top = \begin{bmatrix} U_1^- & U_2 & U_1^+ \end{bmatrix} \begin{bmatrix} -\Sigma_1^- & 0 & 0 \\ 0 & S\Sigma_2 & 0 \\ 0 & 0 & \Sigma_1^+ \end{bmatrix} \begin{bmatrix} (U_1^-)^\top \\ (U_2)^\top \\ (U_1^+)^\top \end{bmatrix}, \tag{3.5}$$

where $U_1^\pm \in \mathbb{R}^{2n\times r^\pm}$, $U_2 \in \mathbb{R}^{2n\times(2n-r^--r^+)}$, $\Sigma_2 \in \mathbb{R}^{(2n-r^--r^+)\times(2n-r^--r^+)}$, $\Sigma_1^\pm \in \mathbb{R}^{r^\pm\times r^\pm}$, and $S = \operatorname{diag}(-I_{n-r^-}, I_{n-r^+})$. We set $\Sigma_1 := \operatorname{diag}(\Sigma_1^-, \Sigma_1^+)$ and $U_1 := \begin{bmatrix} U_1^- & U_1^+ \end{bmatrix}$. Using the reduction matrices

$$W^\top := \Sigma_1^{-\frac{1}{2}} \mathscr{S}_r U_1^\top L \quad \text{and} \quad V := \mathscr{S}_n L^\top U_1 \Sigma_1^{-\frac{1}{2}}, \tag{3.6}$$

we construct the reduced reduced first order model

$$[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}] := [W^\top \mathcal{A} V, W^\top \mathcal{B}, \mathcal{C} V]. \tag{3.7}$$

Next, we state some important properties of the above reduced order model.

THEOREM 3.2. *Let a stabilizable system* $[\mathcal{A},\mathcal{B},\mathcal{C}]$ *be given with* $\mathcal{A}\in\mathbb{R}^{2n\times 2n}$ *and* $\mathcal{B}\in\mathbb{R}^{2n\times m}$ *as in* (3.1) *with* $G\in \mathrm{Gl}_n(\mathbb{R})$, $D\in\mathbb{R}^{n\times n}$ *with* $D=D^\top\geq 0$, *and* $B\in\mathbb{R}^{n\times m}$. *Consider the reduced system* $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}]$ *as constructed as in* (3.5)–(3.7) *for some* $r^+,r^-\in\mathbb{N}$ *which fulfill* (3.2)–(3.4). *Then the following statements are satisfied:*

a) *We have* $\mathscr{W}_{[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}},0]}(\Sigma_1)\leq 0$ *and* $\mathscr{W}_{[\widetilde{\mathcal{A}}^\top,\widetilde{\mathcal{C}}^\top,\widetilde{\mathcal{B}}^\top,0]}(\Sigma_1)\leq 0$. *In particular,* $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}]$ *is passive.*

b) *We have* $\mathrm{diag}(-I_{r^-},I_{r^+})\widetilde{\mathcal{A}}\,\mathrm{diag}(-I_{r^-},I_{r^+}) = \widetilde{\mathcal{A}}^\top$ *and* $\widetilde{\mathcal{B}} = \mathrm{diag}(-I_{r^-},I_{r^+})\widetilde{\mathcal{B}} = \widetilde{\mathcal{C}}^\top$.

c) *The gap metric between the transfer functions* $\mathbf{G}(s)$ *and* $\widetilde{\mathbf{G}}(s)$ *of* $[\mathcal{A},\mathcal{B},\mathcal{C}]$ *and* $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}]$ *can be estimated by*

$$\delta_g(\mathbf{G}(s),\widetilde{\mathbf{G}}(s)) \leq 2\sum_{i=q^-+1}^{h^-}\sigma_i^- + 2\sum_{i=q^++1}^{h^+}\sigma_i^+.$$

d) *If* $\sigma^-_{q^-+1}=0=\sigma^+_{q^++1}$, *then* $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}]$ *is a minimal positive real balanced realization of* $\mathbf{G}(s)$.

Proof. The first two statements can be inferred from the arguments in the proof of [31, Thm. 8], whereas the third part follows from Theorem 2.5 and the last statement from Lemma 2.3. ∎

The exact block structure of the reduced system, as introduced in (1.4), will be part of Section 5. First we need some results from the study of indefinite linear algebra.

4. Preliminaries from indefinite linear algebra. Next, we introduce some notions and results from the study of indefinite linear algebra.

DEFINITION 4.1. *Two pairs* $(S_j,A_j)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}$, $j=1,2$, *consisting of a symmetric matrix* $S_j\in\mathrm{Gl}_n(\mathbb{R})$ *and an* S_j-self-adjoint *matrix* A_j, *i. e.,* $A_j^\top S_j = S_jA_j$, *are called* congruent-similar, *if there exists a* $T\in\mathrm{Gl}_n(\mathbb{R})$ *such that* $T^{-1}A_1T = A_2$ *and* $T^\top S_1 T = S_2$.

In [15], a canonical form under congruence-similarity is given. For the sake if simplicity we will focus on diagonalizable matrices.

THEOREM 4.2. *[15, Sec. I.5, Thm. 5.3] Let* $(S,A)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}$, *where* $S\in\mathrm{Gl}_n(\mathbb{R})$ *is symmetric, and* A *is* S*-self-adjoint and diagonalizable over* $\mathbb{C}$. *Then there exists some* $T\in\mathrm{Gl}_n(\mathbb{R})$, *such that for some* $k,c\in\mathbb{N}$ *with* $2c+k=n$ *and* $\varepsilon_1,\ldots,\varepsilon_k\in\{\pm1\}$, $\lambda_1,\ldots,\lambda_k\in\mathbb{R}$, $\sigma_1,\ldots,\sigma_c\in\mathbb{R}$, $\tau_1,\ldots,\tau_c>0$, *and*

$$\mathscr{J}_1 := \begin{bmatrix}0&1\\1&0\end{bmatrix},\ \mathscr{J}_c := \mathrm{diag}(\underbrace{\mathscr{J}_1,\ldots,\mathscr{J}_1}_{c\ times}),\ \mathscr{P}_{\sigma_i,\tau_i} := \begin{bmatrix}\sigma_i&\tau_i\\-\tau_i&\sigma_i\end{bmatrix},$$

we obtain

$$\begin{aligned} T^\top S T &= \mathrm{diag}(\varepsilon_1,\ldots,\varepsilon_k,\mathscr{J}_c),\\ T^{-1}AT &= \mathrm{diag}(\lambda_1,\ldots,\lambda_k,\mathscr{P}_{\sigma_1,\tau_1},\ldots,\mathscr{P}_{\sigma_c,\tau_c}). \end{aligned} \tag{4.1}$$

It has been further shown in [15, Sec. I.5, Thm. 5.3] that the above is a canonical form for the pair (S, A) under congruence-similarity, if the tuples $(\varepsilon_1, \lambda_1), \dots, (\varepsilon_k, \lambda_k)$ and $(\sigma_1, \tau_1), \dots, (\sigma_c, \tau_c)$ are ordered increasingly with respect to the lexicographical order. It can be seen that the eigenvalues of A in Theorem 4.2 are given by $\lambda_1, \dots, \lambda_k$ and $\sigma_1 \pm \mathrm{i}\tau_1, \dots, \sigma_1 \pm \mathrm{i}\tau_1$. Based on this normal form we derive a special form for S-self-adjoint and diagonalizable matrices whose eigenvalues are in the closed complex left half plane.

Corollary 4.3. *Let $(S, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ be as in Theorem 4.2. If all eigenvalues of A have negative real part, then there exist $T \in \mathrm{Gl}_n(\mathbb{R})$ and $c, k, k_1, k_2 \in \mathbb{N}$ with $k_1 + k_2 = k$ and $2c + k = n$, such that*

$$T^{-1}AT = \begin{bmatrix} 0 & 0 & \mathcal{V} \\ 0 & \Lambda & 0 \\ -\mathcal{V} & 0 & \mathcal{E} \end{bmatrix}, \quad T^\top ST = \operatorname{diag}(-I_{c+k_1}, I_{k_2+c}), \tag{4.2}$$

where $\mathcal{E}, \mathcal{V} \in \mathbb{R}^{c\times c}$ and $\Lambda \in \mathbb{R}^{k\times k}$ are negative definite and diagonal.

Proof. Without loss of generality we can assume that (S, A) is in the canonical form of Theorem 4.2. The assumption on the spectrum of A implies that $\sigma_i > 0$ for $i = 1, \dots, c$ and $\ell = 0$. Since further, $\tau_i > 0$ for all $i = 1, \dots, c$, the matrix

$$\Theta_i := \frac{1}{\sqrt{2\tau_i}} \begin{bmatrix} \sqrt{-\sigma_i + \sqrt{\sigma_i^2 + \tau_i^2}} & -\sqrt{-\sigma_i + \sqrt{\sigma_i^2 + \tau_i^2}} \\ \sqrt{\tau_i}\left(-2\sqrt{-\sigma_i + \sqrt{\sigma_i^2 + \tau_i^2}}\right)^{-1} & \sqrt{\tau_i}\left(-2\sqrt{-\sigma_i + \sqrt{\sigma_i^2 + \tau_i^2}}\right)^{-1} \end{bmatrix} \tag{4.3}$$

is real and straightforward computations show that

$$\Theta_i^\top \mathscr{J}_1 \Theta_i = \mathscr{S}_1 \quad \text{and} \quad \Theta_i^{-1} \mathcal{P}_{\sigma_i,\tau_i} \Theta_i = \mathscr{S}_1 \Theta_i^\top \mathscr{S}_1 \mathcal{P}_{\sigma_i,\tau_i} \Theta_i = \begin{bmatrix} 0 & \nu_i \\ -\nu_i & \eta_i \end{bmatrix},$$

where $\eta_i = \frac{-3\sigma_i^2 + 4\sigma_i\sqrt{\sigma_i^2+\tau_i^2}}{-2\sigma_i + 2\sqrt{\sigma_i^2+\tau_i^2}} < 0$. Setting $T := \operatorname{diag}(I_{2k}, \Theta_1, \dots, \Theta_{c+\ell})$ and suitably interchanging the rows and columns of the tuple $(T^\top ST, T^{-1}AT)$ leads to the form (4.2). ∎

Definition 4.4. *Let $(S, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ be as in Theorem 4.2. Regarding the canonical form in Theorem 4.2, we call the tuple $((\varepsilon_1, \lambda_1), \dots, (\varepsilon_k, \lambda_k))$ the* sign characteristics *of (S, A). Further we call an eigenvalue λ of A an eigenvalue of (S, A), and we say that a real eigenvalue λ of (S, A) is of* positive (negative) *type if $(1, \lambda)$ $((-1, \lambda))$ is contained in the sign characteristics of (S, A).*

We will often indicate that an eigenvalue is of positive (negative) type by equipping it with a superscript "$+$" ("$-$"). Note that λ can be of both negative and positive type. The next result follows rather directly from the definition of the sign characteristics.

Proposition 4.5. *Let $(S, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ be as in Theorem 4.2. An eigenvalue $\lambda \in \mathbb{R}$ of (S, A) is of positive (negative) type, if and only if there exists an eigenvector $v \in \mathbb{R}^n \setminus \{0\}$ of A corresponding to the eigenvalue λ of A such that $v^\top Sv > 0$ $(v^\top Sv < 0)$.*

Proof. This is obviously true, if (S, A) is in the canonical form (4.1). The general statement then follows by a transformation of (S, A) into this canonical form. ∎

We are now able to define the pole and zero sign characteristics of a system $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$. Recall that the zeros of a system $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ are the eigenvalues of the pencil

$\left[\begin{smallmatrix} -sI_n+\mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{smallmatrix}\right]$. If the transfer function is square and invertible, we call a zero μ of $[\mathcal{A},\mathcal{B},\mathcal{C}]$ *semi-simple*, if it is a semi-simple eigenvalue of the latter pencil. That is, the order of the zero μ of $\det\left[\begin{smallmatrix} -sI_n+\mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{smallmatrix}\right]$ equals to the dimension of the kernel of the complex matrix $\left[\begin{smallmatrix} -\mu I_n+\mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{smallmatrix}\right]$.

DEFINITION 4.6. *Let a system $[\mathcal{A},\mathcal{B},\mathcal{C}]$ with invertible transfer function $\mathbf{G}(s) \in \mathbb{R}^{m\times m}$ be given such that, for a signature matrix $S \in \mathrm{Gl}_n(\mathbb{R})$, it holds that $S\mathcal{A}S = \mathcal{A}^\top$ and $\mathcal{B} = S\mathcal{B} = C^\top$. We say that such a system is* internally symmetric *(w.r.t. S). We call the sign characteristics of $(S,\mathcal{A})$ the* pole sign characteristics *of $[\mathcal{A},\mathcal{B},\mathcal{C}]$. Further suppose that the zeros of $[\mathcal{A},\mathcal{B},\mathcal{C}]$ are semi-simple and denote the real zeros of $[\mathcal{A},\mathcal{B},\mathcal{C}]$ by $\mu_1,\dots,\mu_k$. For $i=1,\dots,k$, let $\left[\begin{smallmatrix} v_i \\ w_i \end{smallmatrix}\right] \in \left[\begin{smallmatrix} -\mu_i I_n+\mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{smallmatrix}\right]$. Then we call*

$$\left(\left(-\mathrm{sign}(v_1^\top S v_1),\mu_1\right),\dots,\left(-\mathrm{sign}(v_k^\top S v_k),\mu_k\right)\right)$$

the zero sign characteristics *of $[\mathcal{A},\mathcal{B},\mathcal{C}]$. We say that an eigenvalue $\lambda \in \mathbb{R}$ of $\mathcal{A}$ is a pole of positive (negative) type of $[\mathcal{A},\mathcal{B},\mathcal{C}]$, if $(1,\lambda)$ $((-1,\lambda))$ is contained in the pole sign characteristics of $[\mathcal{A},\mathcal{B},\mathcal{C}]$. Similarly, we say that a zero $\mu \in \mathbb{R}$ of $[\mathcal{A},\mathcal{B},\mathcal{C}]$ is a zero of positive (negative) type of $[\mathcal{A},\mathcal{B},\mathcal{C}]$ if $(1,\mu)$ $((-1,\mu))$ is contained in the zero sign characteristics of $[\mathcal{A},\mathcal{B},\mathcal{C}]$.*

Straightforward calculations show that the system $[\mathcal{A},\mathcal{B},\mathcal{C}]$ with the properties as in Definition 4.6 has a symmetric transfer function. On the other hand, note that the notions of pole and zero sign characteristics are defined for symmetric and invertible transfer functions in [15, Sec. II.3.2] by means of pole and zero sign characteristics of a minimal realization. It is further shown that these are well-defined in the sense that they do not depend on the minimal realization of a given symmetric and invertible transfer function. The basis for this is that, by the results in [15, Sec. II.3.2], for any realization $[\mathcal{A},\mathcal{B},\mathcal{C}]$ of a symmetric and invertible transfer function $\mathbf{G}(s) \in \mathbb{R}(s)^{m\times m}$, there exists a unique nonsingular Hermitian matrix S with

$$S\mathcal{A} = \mathcal{A}^\top S, \qquad S\mathcal{B} = \mathcal{C}^\top, \qquad \mathcal{C} = \mathcal{B}^\top S, \tag{4.4}$$

Further, for two minimal realizations $[\mathcal{A}_i,\mathcal{B}_i,\mathcal{C}_i]$, $i=1,2$ of $\mathbf{G}(s)$, with Hermitian matrices S_1 and S_2 from (4.4), the unique state space transformation $T \in \mathrm{Gl}_n(\mathbb{R})$ between the two realizations fulfills

$$T^{-1} = S_2^{-1} T^\top S_1. \tag{4.5}$$

5. Positive real balanced realizations of second order systems.

The aim in this part is to prove the block structure, introduced in (1.4), of the reduced system from Section 3. For this purpose, we apply three successive transformations to this system. First, we derive a different first order representation of our second order system. Then develop an input-output normal form from which one can read off the different types of system zeros, namely the real and complex ones and those on the imaginary axis. We use this form to arrive at a positive real balanced realization of the original system and deduce that the reduced model has a balanced realization of the same block structure. Later in Section 6, in order to find a second order realization of the reduced system, we actually proceed conversely.

As a first step towards a positive real balanced system we consider a system of the form (3.1) with a special structure which displays the zeros on the imaginary axis.

LEMMA 5.1. *Let a system $[\mathcal{A},\mathcal{B},\mathcal{C}]$ be given with $\mathcal{A} \in \mathbb{R}^{2n\times 2n}$ and $\mathcal{B} \in \mathbb{R}^{2n\times m}$ structured as in (3.1) for some $D \in \mathbb{R}^{n\times n}$ with $D = D^\top$, $G \in \mathrm{Gl}_n(\mathbb{R})$ and $B \in \mathbb{R}^{n\times m}$*

with $\ker B = \{0\}$. *Then the transfer function* $\mathbf{G}(s)$ *of* $[\mathcal{A},\mathcal{B},\mathcal{C}]$ *is invertible. Moreover, there exists a state space transformation* $T = \operatorname{diag}(T_1, T_2) \in \mathrm{Gl}_{2n}(\mathbb{R})$ *with orthogonal* $T_1, T_2 \in \mathrm{Gl}_n(\mathbb{R})$, *such that* $[\mathcal{A}_\mathrm{s},\mathcal{B}_\mathrm{s},\mathcal{C}_\mathrm{s}] = [T^{-1}\mathcal{A}T, T^{-1}\mathcal{B}, \mathcal{C}T]$ *has the form*

$$\mathcal{A}_\mathrm{s} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & G_{31}^\top \\ 0 & 0 & 0 & 0 & G_{22}^\top & G_{32}^\top \\ 0 & 0 & 0 & G_{13}^\top & 0 & G_{33}^\top \\ 0 & 0 & -G_{13} & -D_{11} & 0 & -D_{13} \\ 0 & -G_{22} & 0 & 0 & 0 & 0 \\ -G_{31} & -G_{32} & -G_{33} & -D_{13}^\top & 0 & -D_{33} \end{bmatrix}, \ \mathcal{B}_\mathrm{s} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ B_3 \end{bmatrix} = \mathcal{C}_\mathrm{s}^\top, \tag{5.1}$$

where, for some $\ell \in \mathbb{N}$, *the blocks in the above form are of sizes* m, ℓ, $n-m-\ell$, $n-m-\ell$, ℓ *and* m. *Further,* $G_{11}, B_3 \in \mathbb{R}^{m\times m}$, $G_{22} \in \mathbb{R}^{\ell\times\ell}$ *and* $G_{13} \in \mathbb{R}^{(n-m-\ell)\times(n-m-\ell)}$ *are invertible. All eigenvalues of* $\left[\begin{smallmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11} \end{smallmatrix}\right]$ *have negative real part.*

Moreover, the set of zeros of $[\mathcal{A},\mathcal{B},\mathcal{C}]$ *is given by the union of* $\{0\}$ *and the spectra of* $\left[\begin{smallmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11} \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} 0 & G_{22}^\top \\ -G_{22} & 0 \end{smallmatrix}\right]$. *If further,* $[\mathcal{A},\mathcal{B},\mathcal{C}]$ *has semi-simple zeros, then all zero sign characteristics of* $[\mathcal{A},\mathcal{B},\mathcal{C}]$ *at zero are of positive type, whereas the sign characteristics of the remaining real zeros coincide with the sign characteristics of*

$$\left(\begin{bmatrix} I_{n-m-\ell} & 0 \\ 0 & -I_{n-m-\ell} \end{bmatrix}, \begin{bmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11} \end{bmatrix} \right).$$

Proof. Step 1: We prove the existence of a block diagonal state space transformation $T \in \mathrm{Gl}_{2n}(\mathbb{R})$, such that $[\mathcal{A}_\mathrm{s},\mathcal{B}_\mathrm{s},\mathcal{C}_\mathrm{s}] = [T^{-1}\mathcal{A}T, T^{-1}\mathcal{B}, \mathcal{C}T]$ has the form (5.1) such that B_3, G_{13}, G_{22} and G_{31} are invertible, and $D_{11}v \neq 0$ for each eigenvector $v \in \mathbb{R}^{n-m-\ell} \setminus \{0\}$ of G_{13}.

Since B has full column rank, we have a QR-decomposition $B = T_{21}\left[\begin{smallmatrix} 0 \\ B_3 \end{smallmatrix}\right]$ with invertible B_3. Further, by QR-decomposition of $G^\top T_{21}$ and permutation of rows, we see that there exists some orthogonal $T_{11} \in \mathbb{R}^{n\times n}$ with

$$G^\top T_{21} = T_{11}^\top \begin{bmatrix} 0_{m\times(n-m)} & \widetilde{G}_{21}^\top \\ \widetilde{G}_{12}^\top & \widetilde{G}_{22}^\top \end{bmatrix}.$$

Now applying the state space transformation $\widehat{T}_1 = \operatorname{diag}(T_{11}, T_{21})$ to $[\mathcal{A},\mathcal{B},\mathcal{C}]$ we obtain a system of the form

$$\widehat{T}_1^\top \mathcal{A}\widehat{T}_1 = \begin{bmatrix} 0 & 0 & 0 & \widetilde{G}_{21}^\top \\ 0 & 0 & \widetilde{G}_{12}^\top & \widetilde{G}_{22}^\top \\ 0 & -\widetilde{G}_{12} & -\widetilde{D}_{11} & -\widetilde{D}_{12} \\ -\widetilde{G}_{21} & -\widetilde{G}_{22} & -\widetilde{D}_{12}^\top & -\widetilde{D}_{22} \end{bmatrix}, \quad \widehat{T}_1^\top \mathcal{B} = \big(\mathcal{C}\widehat{T}_1\big)^\top = \begin{bmatrix} 0 \\ 0 \\ 0 \\ B_3 \end{bmatrix}.$$

Our next step is to apply a further state space transformation which separates the purely imaginary eigenvalues of $\left[\begin{smallmatrix} 0 & \widetilde{G}_{12}^\top \\ -\widetilde{G}_{12} & -\widetilde{D}_{11} \end{smallmatrix}\right]$ from those with negative real part. To this end, we perform a singular value decomposition $\widetilde{G}_{12} = T_{12}\widehat{G}_{12}T_{22}^\top$ with orthogonal matrices $T_{12}, T_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$ and a diagonal matrix $\widehat{G}_{12} \in \mathbb{R}^{(n-m)\times(n-m)}$. Further, for each eigenspace of $\widehat{G}_{12}$, we perform an orthogonal decomposition into the intersection of this eigenspace with $\ker T_{22}^\top \widetilde{D}_{11} T_{22}$ and its orthogonal complement in this eigenspace. An accordant orthogonal transformation along with a permutation matrix leads to the existence of some orthogonal matrices $T_{23}, T_{13} \in \mathbb{R}^{(n-m)\times(n-m)}$

that lead to the following transformed matrices: First, for some $\ell \in \mathbb{N}$, $G_{22} \in \mathbb{R}^{\ell\times\ell}$, $G_{13} \in \mathbb{R}^{(n-m-\ell)\times(n-m-\ell)}$ we have

$$T_{23}^\top \widetilde{G}_{12} T_{13} = \begin{bmatrix} 0 & G_{13} \\ G_{22} & 0 \end{bmatrix}.$$

Moreover, for some $D_{11} \in \mathbb{R}^{(n-m-\ell)\times(n-m-\ell)}$, $D_{13} \in \mathbb{R}^{(n-m-\ell)\times m}$, $D_{33} \in \mathbb{R}^{m\times m}$ it holds that

$$\begin{bmatrix} T_{23}^\top T_{22}^\top & 0 \\ 0 & I_m \end{bmatrix} T_{21}^\top D T_{21} \begin{bmatrix} T_{22}T_{23} & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} D_{11} & 0 & D_{13} \\ 0 & 0 & 0 \\ D_{13}^\top & 0 & D_{33} \end{bmatrix}.$$

Last, for each eigenvector $v \in \mathbb{R}^{n-m-\ell} \setminus \{0\}$ of the diagonal matrix G_{13} we have $D_{11}v \neq 0$ and for some $G_{31} \in \mathbb{R}^{m\times m}$, $G_{32} \in \mathbb{R}^{m\times(n-m-\ell)}$, $G_{33} \in \mathbb{R}^{m\times(n-m-\ell)}$ it holds that

$$\begin{bmatrix} T_{23}^\top T_{22}^\top & 0 \\ 0 & I_m \end{bmatrix} T_{21}^\top G T_{11} \begin{bmatrix} I_m & 0 \\ 0 & T_{12}T_{13} \end{bmatrix} = \begin{bmatrix} 0 & 0 & G_{13} \\ 0 & G_{22} & 0 \\ G_{31} & G_{32} & G_{33} \end{bmatrix}.$$

In particular, G_{13}, G_{22} and G_{31} are invertible since G is invertible. Altogether, for the orthogonal matrices $T_1 = T_{11}\left[\begin{smallmatrix} I_m & 0 \\ 0 & T_{12}T_{13} \end{smallmatrix}\right]$, $T_2 = T_{21}\left[\begin{smallmatrix} T_{22}T_{23} & 0 \\ 0 & I_m \end{smallmatrix}\right]$, a state space transformation with $T = \operatorname{diag}(T_1, T_2) \in \mathrm{Gl}_{2n}(\mathbb{R})$ results into a system $[\mathcal{A}_\mathrm{s}, \mathcal{B}_\mathrm{s}, \mathcal{C}_\mathrm{s}] = [T^{-1}\mathcal{A}T, T^{-1}\mathcal{B}, \mathcal{C}T]$ which is of the form (5.1).

Step 2: We prove that for the construction in Step 1 all eigenvalues of $\left[\begin{smallmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11} \end{smallmatrix}\right]$ have negative real part.

The fact that all eigenvalues of $\left[\begin{smallmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11} \end{smallmatrix}\right]$ have nonpositive real part follows from the fact that the sum of this matrix and its Hermitian is negative semidefinite. To show that it does not have any eigenvalues on the imaginary axis, assume that $\omega \in \mathbb{R}$, $v_1, v_2 \in \mathbb{C}^{n-m-\ell}$ are given such that

$$\begin{bmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11} \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathrm{i}\omega \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{5.2}$$

A multiplication of (5.2) from the right with $\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)^*$ and taking the real part yields $v_2^* D_{11} v_2 = 0$, whence, by $D_{11} \geq 0$, $D_{11}v_2 = 0$. Hence

$$\begin{bmatrix} 0 & G_{13}^\top \\ -G_{13} & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathrm{i}\omega \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \ \wedge\ D_{11}v_2 = 0. \tag{5.3}$$

On the other hand, the first relation in (5.3) yields $G_{13}G_{13}^\top v_2 = \omega^2 v_2$, whence, by the fact that G_{13} is diagonal, v_2 is an eigenvector of G_{13}. Then we obtain $v_2 = 0$ by the results of Step 1, and the invertibility of G_{13} further gives rise to $v_1 = 0$.

Step 3: We show that the transfer function $\mathbf{G}(s)$ is invertible, and the set of zeros of $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ is given by the union of $\{0\}$ and the spectra of $\left[\begin{smallmatrix} 0 & G_{13}^\top \\ -G_{31} & -D_{11} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & G_{22}^\top \\ -G_{22} & 0 \end{smallmatrix}\right]$.

This follows by the fact that

$$\det \begin{bmatrix} -sI_{2n} + \mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{bmatrix} = c \cdot s^m \cdot \det \begin{bmatrix} sI_{n-m-\ell} & -G_{13}^\top \\ G_{13} & sI_{n-m-\ell} + D_{11} \end{bmatrix} \cdot \det \begin{bmatrix} sI_\ell & -G_{22}^\top \\ G_{22} & sI_\ell \end{bmatrix}$$

for some $c \in \mathbb{R} \setminus \{0\}$.

Step 4: We prove the statement about the sign characteristics.

This follows, since $\left[\begin{smallmatrix} v_3 \\ v_4 \end{smallmatrix}\right] \in \mathbb{C}^{2(n-m-\ell)}$ is an eigenvector of $\left[\begin{smallmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11} \end{smallmatrix}\right]$ corresponding to the eigenvalue $\lambda \in \mathbb{C}$, if and only if there exists some $v_7 \in \mathbb{C}^m$ such that

$$\begin{bmatrix} 0 \\ 0 \\ v_3 \\ v_4 \\ 0 \\ 0 \\ v_7 \end{bmatrix} \in \ker \begin{bmatrix} -\lambda I_{2n} + \mathcal{A}_\mathrm{s} & \mathcal{B}_\mathrm{s} \\ \mathcal{C}_\mathrm{s} & 0 \end{bmatrix}.$$

The statement for the sign characteristics of the zeros at zero is completely analogous.∎

Starting from the second order system structured as in the above lemma, we can determine a normal form which displays the different type of zeros. In particular, we find that solutions of the KYP inequalities from the positive real lemma are block diagonal matrices structured accordingly to the zero blocks. This helps us to derive a positive real balanced system of the form (1.4). Moreover, in Section 6, we take this normal form as a basis to bring the reduced system back to second order form and to check a necessary condition whether this is actually possible. Before we present the normal form we need a small lemma.

Lemma 5.2. *Let $Y \in \mathbb{R}^{n\times n}$ be skew-symmetric and $Z \in \mathbb{R}^{n\times n}$ be symmetric and positive semidefinite. Then $ZY + Y^\top Z \leq 0$, if and only if $ZY + Y^\top Z = 0$.*

Proof. Since by using $Z = Z^\top$ and $Y = -Y^\top$, an evaluation of the diagonal entries of $ZY + Y^\top Z$ yields that these vanish, $ZY + Y^\top Z \leq 0$ implies that $ZY + Y^\top Z = 0$. ∎

Theorem 5.3. *Let a stabilizable system $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ be given. Assume that $\mathcal{A} \in \mathbb{R}^{2n\times 2n}$ and $\mathcal{B}, \mathcal{C}^\top \in \mathbb{R}^{2n\times m}$ are structured as in (3.1) for some $G, D \in \mathbb{R}^{n\times n}$ with $D = D^\top \geq 0$, $G \in \mathrm{Gl}_n(\mathbb{R})$ and $B \in \mathbb{R}^{n\times m}$. Suppose that the zeros of the system are semi-simple. Then there exists a state space transformation $T \in \mathrm{Gl}_{2n}(\mathbb{R})$ such that the system $[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}] := [T^{-1}\mathcal{A}T, T^{-1}\mathcal{B}, \mathcal{C}T]$ has the block form*

(5.4)
$$\mathcal{A}_\mathrm{n} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_{27} & \mathcal{A}_{28} \\ 0 & 0 & 0 & 0 & 0 & \mathcal{A}_{36} & 0 & \mathcal{A}_{38} \\ 0 & 0 & 0 & \mathcal{A}_{44} & 0 & 0 & 0 & \mathcal{A}_{48} \\ 0 & 0 & 0 & 0 & \mathcal{A}_{55} & 0 & 0 & \mathcal{A}_{58} \\ 0 & 0 & -\mathcal{A}_{36}^\top & 0 & 0 & \mathcal{A}_{66} & 0 & \mathcal{A}_{68} \\ 0 & -\mathcal{A}_{27}^\top & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathcal{A}_{18}^\top & -\mathcal{A}_{28}^\top & -\mathcal{A}_{38}^\top & -\mathcal{A}_{48}^\top & \mathcal{A}_{58}^\top & \mathcal{A}_{68}^\top & 0 & \mathcal{A}_{88} \end{bmatrix}, \quad \mathcal{C}_\mathrm{n}^\top = \mathcal{B}_\mathrm{n} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{B}_8 \end{bmatrix},$$

where

a) $\mathcal{A}_{18}, \mathcal{B}_8 \in \mathrm{Gl}_m(\mathbb{R})$, $\mathcal{A}_{27} \in \mathrm{Gl}_\ell(\mathbb{R})$, $\mathcal{A}_{36} \in \mathrm{Gl}_c(\mathbb{R})$ and $\mathcal{A}_{66} < 0$;

b) $\mathcal{A}_{44} = \mathrm{diag}(\mu_1^+, \ldots, \mu_k^+)$ and $\mathcal{A}_{55} = \mathrm{diag}(\mu_1^-, \ldots, \mu_k^-)$, with $\mu_1^\pm \leq \ldots \leq \mu_k^\pm < 0$. If $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ is minimal, then the μ_i^+ and the μ_i^- are the negative (real and nonzero) zeros of $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ of positive and negative type, respectively;

c) $n = k + c + \ell + m$ and $2\ell + m$ is the number of zeros of $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ on $\mathrm{i}\mathbb{R}$ counted with multiplicities.

Further, all solutions $P \geq 0$, $Q \geq 0$ of the KYP inequalities $\mathscr{W}_{[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}, 0]}(P) \leq 0$ and $\mathscr{W}_{[\mathcal{A}_\mathrm{n}^\top, \mathcal{C}_\mathrm{n}^\top, \mathcal{B}_\mathrm{n}^\top, 0]}(Q) \leq 0$ have the block form $P = \mathrm{diag}(I_{m+\ell}, P_2, I_{m+\ell})$ and $Q = \mathrm{diag}(I_{m+\ell}, Q_2, I_{m+\ell})$ for some $P_2, Q_2 \in \mathbb{R}^{2(c+k)\times 2(c+k)}$.

Proof. Step 1: We show that there exists a state space transformation $\widetilde{T} \in \mathrm{Gl}_{2n}(\mathbb{R})$ such that $\widetilde{T}^\top \mathscr{S}_n \widetilde{T} = \mathscr{S}_n$ and $[\widetilde{\mathcal{A}}_\mathrm{n}, \widetilde{\mathcal{B}}_\mathrm{n}, \widetilde{\mathcal{C}}_\mathrm{n}] := [\widetilde{T}^{-1}\mathcal{A}\widetilde{T}, \widetilde{T}^{-1}\mathcal{B}, \mathcal{C}\widetilde{T}]$ has the form (5.4). Moreover, we show that this realization fulfills a)–c) in the above theorem.

Without loss of generality to assume that the system is already in the form $[\mathcal{A}, \mathcal{B}, \mathcal{C}] = [\mathcal{A}_\mathrm{s}, \mathcal{B}_\mathrm{s}, \mathcal{C}_\mathrm{s}]$ as in Lemma 5.1. In particular, all eigenvalues of $\widehat{\mathcal{A}}_\mathrm{s} := \begin{bmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11} \end{bmatrix} \in \mathbb{R}^{2(n-m-\ell)\times 2(n-m-\ell)}$ have negative real part. Since the non-real eigenvalues occur in pairs and $\widehat{\mathcal{A}}_\mathrm{s} \in \mathbb{R}^{2(n-m-\ell)\times 2(n-m-\ell)}$, there exists some $c \in \mathbb{N}_0$ such that $2c$ is the number of non-real eigenvalues (counted with multiplicities). Hence, the number of real eigenvalues of $\widehat{\mathcal{A}}_\mathrm{s}$ (again counted with multiplicities) is $2k$ for $k = n - c - \ell - m$. Now, according to Corollary 4.3, k real eigenvalues $\mu_1^+ \leq \ldots \leq \mu_k^+ < 0$ of $(\mathscr{S}_{n-m}, \widehat{\mathcal{A}}_\mathrm{s})$ are of positive type, whereas the remaining k real eigenvalues $\mu_1^- \leq \ldots \leq \mu_k^- < 0$ are of negative type. This corollary further implies that there exists some $\widehat{T} \in \mathrm{Gl}_{2(n-m)}$ with $\widehat{T}^\top \mathscr{S}_{n-m} \widehat{T} = \mathscr{S}_{n-m}$ and

$$\widehat{T}^{-1}\widehat{\mathcal{A}}_\mathrm{s}\widehat{T} = \begin{bmatrix} 0 & 0 & 0 & \mathcal{A}_{36} \\ 0 & \mathcal{A}_{44} & 0 & 0 \\ 0 & 0 & \mathcal{A}_{55} & 0 \\ -\mathcal{A}_{36}^\top & 0 & 0 & \mathcal{A}_{66} \end{bmatrix},$$

where, for some $\eta_i < 0$, $\nu_i, \in \mathbb{R}\backslash\{0\}$ for $i = 1, \ldots, c$ we have

$$\begin{aligned} \mathcal{A}_{44} &= \mathrm{diag}(\mu_1^+, \ldots, \mu_k^+) \in \mathrm{Gl}_k(\mathbb{R}), & \mathcal{A}_{55} &= \mathrm{diag}(\mu_1^-, \ldots, \mu_k^-) \in \mathrm{Gl}_k(\mathbb{R}), \\ \mathcal{A}_{66} &= \mathrm{diag}(\eta_1, \ldots, \eta_c) \in \mathrm{Gl}_c(\mathbb{R}), & \mathcal{A}_{36} &= \mathrm{diag}(\nu_1, \ldots, \nu_c) \in \mathrm{Gl}_k(\mathbb{R}). \end{aligned}$$

Then $\widetilde{T} := \mathrm{diag}(I_{m+\ell}, \widehat{T}, I_{m+\ell})$ fulfills $\widetilde{T}^\top \mathscr{S}_n \widetilde{T} = \mathscr{S}_n$, and an application of this state space transformation leads to a realization $[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}]$ of the desired form. Further, since $\mathcal{A}_{18} = G_{31}^\top$, $\mathcal{A}_{27} = G_{22}^\top$ and $\mathcal{B}_8 = B_3$, those matrices are invertible by Lemma 5.1.

Step 2: Suppose that $P \geq 0$ solves $\mathscr{W}_{[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}, 0]}(P) \leq 0$ and partition $P = (P_{ij})_{i,j=1,\ldots,8}$ according to the block structure of $\mathcal{A}_\mathrm{n}$. We show that $P_{11} = P_{88} = I_m$ and $P_{1i} = 0$, $P_{j8} = 0$ for $i = 2, \ldots, 8$ and $j = 1, \ldots, 7$.

Suppose $P \geq 0$ solves the KYP inequality. Then $P\mathcal{B}_\mathrm{n} - \mathcal{C}_\mathrm{n}^\top = 0$ together with $\mathcal{B}_8 \in \mathrm{Gl}_m(\mathbb{R})$ implies that $P_{j8} = 0$, for $j = 1, \ldots, 7$ and $P_{88} = I_m$. This implies that the upper left block of size $m \times m$ of the left-hand side of $\mathcal{A}_\mathrm{n}^\top P + P\mathcal{A}_\mathrm{n} \leq 0$ is zero, whence all the corresponding off-diagonal blocks of $\mathcal{A}_\mathrm{n}^\top P + P\mathcal{A}_\mathrm{n}$ have to be zero as well. Equivalently, $-\mathcal{A}_{27}P_{17}^\top = 0$, $-\mathcal{A}_{36}P_{16}^\top = 0$, $\mathcal{A}_{44}P_{14}^\top = 0$, $\mathcal{A}_{55}P_{15}^\top = 0$, $\mathcal{A}_{36}^\top P_{13}^\top + \mathcal{A}_{66}P_{16}^\top = 0$ and $\mathcal{A}_{27}^\top P_{12}^\top = 0$. Since $\mathcal{A}_{27}$, $\mathcal{A}_{36}$, $\mathcal{A}_{44}$, $\mathcal{A}_{55}$ and $\mathcal{A}_{66}$ are invertible, $P_{1i} = 0$ for $i = 2, \ldots, 8$.

Step 3: We show that $P_{2i} = 0$ and $P_{7i} = 0$ for $i = 3, \ldots, 6$.

We have that

$$\begin{bmatrix} 0 & -\mathcal{A}_{27} \\ \mathcal{A}_{27}^\top & 0 \end{bmatrix} \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} + \underbrace{\begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix}}_{=:Z} \underbrace{\begin{bmatrix} 0 & \mathcal{A}_{27} \\ -\mathcal{A}_{27}^\top & 0 \end{bmatrix}}_{=:Y} \leq 0, \tag{5.5}$$

where, due to Lemma 5.2, equality holds. We set

$$\breve{\mathcal{A}} := \begin{bmatrix} 0 & 0 & 0 & \mathcal{A}_{36} \\ 0 & \mathcal{A}_{44} & 0 & 0 \\ 0 & 0 & \mathcal{A}_{55} & 0 \\ -\mathcal{A}_{36}^\top & 0 & 0 & \mathcal{A}_{66} \end{bmatrix}, \; \breve{P}_1 := \begin{bmatrix} P_{33} & \cdots & P_{36} \\ \vdots & & \vdots \\ P_{36}^\top & \cdots & P_{66} \end{bmatrix}, \; \breve{P}_2 := \begin{bmatrix} P_{23}^\top & P_{37} \\ P_{24}^\top & P_{47} \\ P_{25}^\top & P_{57} \\ P_{26}^\top & P_{67} \end{bmatrix},$$

By considering the principal submatrix of $\mathcal{A}_{\mathrm{n}}^\top P + P\mathcal{A}_{\mathrm{n}}$ obtained by removing the first and last m rows, we obtain

$$0 \geq \begin{bmatrix} \breve{\mathcal{A}} & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} \breve{P}_1 & \breve{P}_2 \\ \breve{P}_2^\top & Z \end{bmatrix} + \begin{bmatrix} \breve{P}_1 & \breve{P}_2 \\ \breve{P}_2^\top & Z \end{bmatrix} \begin{bmatrix} \breve{\mathcal{A}}^\top & 0 \\ 0 & Y^\top \end{bmatrix} = \begin{bmatrix} \breve{\mathcal{A}}^\top \breve{P}_1 + \breve{P}_1 \breve{\mathcal{A}} & \breve{\mathcal{A}}\breve{P}_2 + \breve{P}_2 Y^\top \\ Y\breve{P}_2 + \breve{P}_2\breve{\mathcal{A}}^\top & 0 \end{bmatrix},$$

with Z, Y as in (5.5). Thus, we obtain the Sylvester equation $Y\breve{P}_2^\top + \breve{P}_2^\top \breve{\mathcal{A}}^\top = 0$ for $\breve{P}_2$. The spectrum of Y is contained on the imaginary axis, and all eigenvalues of $\breve{\mathcal{A}}^\top$ have negative real part. Since Y and $-\breve{\mathcal{A}}^\top$ have no common eigenvalues, we obtain from [20, Thm. 2.4.4.1] that $\breve{P}_2 = 0$.

Step 4: We show that $\left(\left[\begin{smallmatrix} 0 & -\mathcal{A}_{27} \\ \mathcal{A}_{27}^\top & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \mathcal{A}_{28} \\ 0 \end{smallmatrix}\right] \right)$ is controllable.

Assume that $\lambda \in \mathbb{C}$ and $v_2, v_7 \in \mathbb{C}^\ell$, such that

$$\begin{bmatrix} v_2^* & v_7^* \end{bmatrix} \begin{bmatrix} \lambda I_\ell & \mathcal{A}_{27} & -\mathcal{A}_{28} \\ -\mathcal{A}_{27}^\top & \lambda I_\ell & 0 \end{bmatrix} = 0.$$

Since $\left[\begin{smallmatrix} 0 & -\mathcal{A}_{27} \\ \mathcal{A}_{27}^\top & 0 \end{smallmatrix}\right]$ is skew-symmetric, it follows that $\lambda = \mathrm{i}\omega$ for some $\omega \in \mathbb{R}$. By setting $v := \left[\begin{smallmatrix} 0 & v_2^* & 0 & 0 & 0 & 0 & v_7^* & 0 \end{smallmatrix}\right]^* \in \mathbb{C}^{2n}$ with block structure according to that of $\mathcal{A}_{\mathrm{n}}$, we now obtain that $v^* \begin{bmatrix} \mathrm{i}\omega I_{2n} - \mathcal{A}_{\mathrm{n}} & \mathcal{B}_{\mathrm{n}} \end{bmatrix} = 0$. Since $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ is assumed to be stabilizable, $[\mathcal{A}_{\mathrm{n}}, \mathcal{B}_{\mathrm{n}}, \mathcal{C}_{\mathrm{n}}]$ is stabilizable as well. This leads to $v = 0$, whence $v_2 = v_7 = 0$.

Step 5: Let $P \geq 0$, $Q \geq 0$ be solutions of the KYP inequalities $\mathscr{W}_{[\mathcal{A}_{\mathrm{n}}, \mathcal{B}_{\mathrm{n}}, \mathcal{C}_{\mathrm{n}}, 0]}(P) \leq 0$ and $\mathscr{W}_{[\mathcal{A}_{\mathrm{n}}^\top, \mathcal{C}_{\mathrm{n}}^\top, \mathcal{B}_{\mathrm{n}}^\top, 0]}(Q) \leq 0$. We show that $P = \operatorname{diag}(I_{m+\ell}, P_2, I_{m+\ell})$ and $Q = \operatorname{diag}(I_{m+\ell}, Q_2, I_{m+\ell})$ for some $P_2, Q_2 \in \mathbb{R}^{2(c+k)\times 2(c+k)}$.

Since the solutions $P \geq 0$ fulfills $\mathscr{W}_{[\mathcal{A}_{\mathrm{n}}, \mathcal{B}_{\mathrm{n}}, \mathcal{C}_{\mathrm{n}}, 0]}(P) \leq 0$, if and only if $Q = \mathscr{S}_n P \mathscr{S}_n \geq 0$ fulfills $\mathscr{W}_{[\mathcal{A}_{\mathrm{n}}^\top, \mathcal{C}_{\mathrm{n}}^\top, \mathcal{B}_{\mathrm{n}}^\top, 0]}(Q) \leq 0$, it suffices to prove the statement only for $P \geq 0$ with $\mathscr{W}_{[\mathcal{A}_{\mathrm{n}}, \mathcal{B}_{\mathrm{n}}, \mathcal{C}_{\mathrm{n}}, 0]}(P) \leq 0$. We partition $P = (P_{ij})_{i,j=1,\dots,8}$ according to the block structure of $\mathcal{A}_{\mathrm{n}}$. As $[\mathcal{A}_{\mathrm{n}}, \mathcal{B}_{\mathrm{n}}, \mathcal{C}_{\mathrm{n}}]$ is structured as the system $[\widetilde{\mathcal{A}}_{\mathrm{n}}, \widetilde{\mathcal{B}}_{\mathrm{n}}, \widetilde{\mathcal{C}}_{\mathrm{n}}]$, we can use our findings in Step 2 and Step 3 to see that $P_{11} = P_{88} = I_m$ and $P_{1i} = 0$, $P_{j8} = 0$ for $i = 2, \dots, 8$ and $j = 1, \dots, 7$; and $P_{2i} = 0$ and $P_{7i} = 0$ for $i = 3, \dots, 6$. Now a straightforward calculation now yields that

(5.6)
$$\begin{bmatrix} 0 & -\mathcal{A}_{27} & -\mathcal{A}_{28} \\ \mathcal{A}_{27}^\top & 0 & 0 \\ \mathcal{A}_{28}^\top & 0 & \mathcal{A}_{88} \end{bmatrix} \begin{bmatrix} P_{22} & P_{27} & 0 \\ P_{27}^\top & P_{77} & 0 \\ 0 & 0 & I_m \end{bmatrix} + \begin{bmatrix} P_{22} & P_{27} & 0 \\ P_{27}^\top & P_{77} & 0 \\ 0 & 0 & I_m \end{bmatrix} \begin{bmatrix} 0 & \mathcal{A}_{27} & \mathcal{A}_{28} \\ -\mathcal{A}_{27}^\top & 0 & 0 \\ -\mathcal{A}_{28}^\top & 0 & \mathcal{A}_{88} \end{bmatrix} \leq 0.$$

An evaluation of the upper right two block gives

$$\begin{bmatrix} 0 & -\mathcal{A}_{27} \\ \mathcal{A}_{27}^\top & 0 \end{bmatrix} \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} + \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} \begin{bmatrix} 0 & \mathcal{A}_{27} \\ -\mathcal{A}_{27}^\top & 0 \end{bmatrix} \leq 0,$$

whence, by Lemma 5.2, the latter inequality becomes an equality. Invoking this, an evaluation of the blocks "31" and "32" leads to

$$\begin{bmatrix} \mathcal{A}_{28}^\top & 0 \end{bmatrix} \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{28}^\top & 0 \end{bmatrix}.$$

This altogether yields

$$\begin{aligned} \begin{bmatrix} 0 & \mathcal{A}_{27} \\ -\mathcal{A}_{27}^\top & 0 \end{bmatrix} \left(I_{2\ell} - \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} \right) &= \left(I_{2\ell} - \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} \right) \begin{bmatrix} 0 & \mathcal{A}_{27} \\ -\mathcal{A}_{27}^\top & 0 \end{bmatrix}, \\ \begin{bmatrix} \mathcal{A}_{28}^\top & 0 \end{bmatrix} \left(I_{2\ell} - \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} \right) &= 0. \end{aligned}$$

Hence, $\operatorname{im}\left(I_{2\ell}-\left[\begin{smallmatrix}P_{22} & P_{27}\\ P_{27}^\top & P_{77}\end{smallmatrix}\right]\right)$ is an $\left[\begin{smallmatrix}0 & \mathcal{A}_{27}\\ -\mathcal{A}_{27}^\top & 0\end{smallmatrix}\right]$-invariant subspace and is contained in $\ker\begin{bmatrix}\mathcal{A}_{28}^\top & 0\end{bmatrix}$. However, by Step 4, $\left(\left[\begin{smallmatrix}0 & -\mathcal{A}_{27}\\ \mathcal{A}_{27}^\top & 0\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathcal{A}_{28}\\ 0\end{smallmatrix}\right]\right)$ is controllable, thus

$$\operatorname{im}\left(I_{2\ell}-\begin{bmatrix}P_{22} & P_{27}\\ P_{27}^\top & P_{77}\end{bmatrix}\right)=\{0\}$$

and $P_{22}=P_{77}=I_\ell$ and $P_{27}=0$. ∎

With the previous theorem at hand and the normal form therein we can now exploit the structure of the original system and the solutions of the KYP inequalities to find a positive real balanced realization of the original system of the form (1.4).

THEOREM 5.4. *Let a stabilizable system $[\mathcal{A},\mathcal{B},\mathcal{C}]$ be given with transfer function $\mathbf{G}(s)$. Assume that $\mathcal{A}\in\mathbb{R}^{2n\times 2n}$ and $\mathcal{B},\mathcal{C}^\top\in\mathbb{R}^{2n\times m}$ are structured as in (3.1) for some $G,D\in\mathbb{R}^{n\times n}$ with $D=D^\top\geq 0$, $G\in\mathrm{Gl}_n(\mathbb{R})$ and $B\in\mathbb{R}^{n\times m}$ with $\ker B=\{0\}$. Suppose that the system has semi-simple zeros. Then $\mathbf{G}(s)$ has a positive real balanced realization $[\mathcal{A}_\mathrm{b},\mathcal{B}_\mathrm{b},\mathcal{C}_\mathrm{b}]$ of the block form*

$$\mathcal{A}_\mathrm{b}=\begin{bmatrix}0&0&0&0&0&\widehat{\mathcal{A}}_{16}\\ 0&0&0&0&\widehat{\mathcal{A}}_{25}&\widehat{\mathcal{A}}_{26}\\ 0&0&\widehat{\mathcal{A}}_{33}&\widehat{\mathcal{A}}_{34}&0&\widehat{\mathcal{A}}_{36}\\ 0&0&-\widehat{\mathcal{A}}_{34}^\top&\widehat{\mathcal{A}}_{44}&0&\widehat{\mathcal{A}}_{46}\\ 0&-\widehat{\mathcal{A}}_{25}^\top&0&0&0&0\\ -\widehat{\mathcal{A}}_{16}^\top&-\widehat{\mathcal{A}}_{26}^\top&-\widehat{\mathcal{A}}_{36}^\top&\widehat{\mathcal{A}}_{46}^\top&0&\widehat{\mathcal{A}}_{66}\end{bmatrix},\quad \mathcal{B}_\mathrm{b}=\begin{bmatrix}0\\0\\0\\0\\0\\ \widehat{\mathcal{B}}_6\end{bmatrix}=\mathcal{C}_\mathrm{b}^\top, \tag{5.7}$$

where $\widehat{\mathcal{A}}_{16},\widehat{\mathcal{B}}_6\in\mathrm{Gl}_m(\mathbb{R})$, $\widehat{\mathcal{A}}_{66}\in\mathbb{R}^{m\times m}$, $\widehat{\mathcal{A}}_{33}\in\mathbb{R}^{p_1\times p_1}$, $\widehat{\mathcal{A}}_{44}\in\mathbb{R}^{p_2\times p_2}$, $\widehat{\mathcal{A}}_{25}\in\mathbb{R}^{\ell\times\ell}$ and $\widehat{n}=2m+p_1+p_2+2\ell$ is the order of the minimal system. Further, the minimal solutions of the KYP inequalities $\mathscr{W}_{[\mathcal{A}_\mathrm{b},\mathcal{B}_\mathrm{b},\mathcal{C}_\mathrm{b},0]}(P)\leq 0$ and $\mathscr{W}_{[\mathcal{A}_\mathrm{b}^\top,\mathcal{C}_\mathrm{b}^\top,\mathcal{B}_\mathrm{b}^\top,0]}(Q)\leq 0$ are given by $P=Q=\operatorname{diag}(I_{m+\ell},\Pi,I_{m+\ell})$ for some diagonal matrix $\Pi\in\mathbb{R}^{(p_1+p_2)\times(p_1+p_2)}$. Moreover, the system

$$\left[\begin{bmatrix}\widehat{\mathcal{A}}_{33}&\widehat{\mathcal{A}}_{34}\\ -\widehat{\mathcal{A}}_{34}^\top&\widehat{\mathcal{A}}_{44}\end{bmatrix},\begin{bmatrix}\widehat{\mathcal{A}}_{36}\\ \widehat{\mathcal{A}}_{46}\end{bmatrix},\begin{bmatrix}\widehat{\mathcal{A}}_{36}^\top & -\widehat{\mathcal{A}}_{46}^\top\end{bmatrix},-\widehat{\mathcal{A}}_{66}\right]=:[\mathcal{A}_\mathrm{z},\mathcal{B}_\mathrm{z},\mathcal{C}_\mathrm{z},\mathcal{D}_\mathrm{z}] \tag{5.8}$$

is asymptotically stable and positive real balanced, where Π is the minimal solution of $\mathscr{W}_{[\mathcal{A}_\mathrm{z},\mathcal{B}_\mathrm{z},\mathcal{C}_\mathrm{z},\mathcal{D}_\mathrm{z}]}(\widehat{P})\leq 0$ and $\mathscr{W}_{[\mathcal{A}_\mathrm{z}^\top,\mathcal{C}_\mathrm{z}^\top,\mathcal{B}_\mathrm{z}^\top,\mathcal{D}_\mathrm{z}^\top]}(\widehat{Q})\leq 0$ and the spectrum of $\mathcal{A}_\mathrm{z}$ coincides with the set of zeros of $[\mathcal{A}_\mathrm{b},\mathcal{B}_\mathrm{b},\mathcal{C}_\mathrm{b}]$ with negative real part.

Proof. Theorem 5.3 provides that $\mathbf{G}(s)$ has a realization $[\mathcal{A}_\mathrm{n},\mathcal{B}_\mathrm{n},\mathcal{C}_\mathrm{n}]$ of the block form (5.4), where $P_{\min}$ and $Q_{\min}$ have the block structure $P_{\min}=\operatorname{diag}(I_{m+\ell},P_2,I_{\ell+m})$ and $Q_{\min}=\operatorname{diag}(I_{m+\ell},Q_2,I_{\ell+m})$ for some matrices $P_2,Q_2>0$. Let $[\mathcal{A}_\mathrm{b},\mathcal{B}_\mathrm{b},\mathcal{C}_\mathrm{b}]$ be the reduced system of $[\mathcal{A}_\mathrm{n},\mathcal{B}_\mathrm{n},\mathcal{C}_\mathrm{n}]$ constructed as in (3.7) for the choice $r^-=n-\dim\ker\Sigma^-$ and $r^+=n-\dim\ker\Sigma^+$. Here, the matrices $W^\top$ and V from (3.7) have the same block structure as $P_{\min}$, namely $W^\top=\operatorname{diag}(I_{m+\ell},W_2^\top,I_{m+\ell})$ and $V=\operatorname{diag}(I_{m+\ell},V_2,I_{m+\ell})$ for some matrices $W_2,V_2\in\mathbb{R}^{(p_1+p_2)\times(p_1+p_2)}$. Inspecting the proof of Theorem 3.2, its statements are also valid for the reduced system $[\mathcal{A}_\mathrm{b},\mathcal{B}_\mathrm{b},\mathcal{C}_\mathrm{b}]$. Thus, by Theorem 3.2 d) and c) it follows that $[\mathcal{A}_\mathrm{b},\mathcal{B}_\mathrm{b},\mathcal{C}_\mathrm{b}]$ is a minimal, positive real balanced realization of $\mathbf{G}(s)$ and $\operatorname{diag}(-I_{r^-},I_{r^+})\mathcal{A}_\mathrm{b}\operatorname{diag}(-I_{r^-},I_{r^+})=\mathcal{A}_\mathrm{b}^\top$. Altogether, the realization $[\mathcal{A}_\mathrm{b},\mathcal{B}_\mathrm{b},\mathcal{C}_\mathrm{b}]$ has the block form (5.7). Moreover, Theorem 5.3 and the block form of the reduction matrices V and W provide that all solutions of the KYP inequalities $\mathscr{W}_{[\mathcal{A}_\mathrm{b},\mathcal{B}_\mathrm{b},\mathcal{C}_\mathrm{b},0]}(P)\leq 0$ and $\mathscr{W}_{[\mathcal{A}_\mathrm{b}^\top,\mathcal{C}_\mathrm{b}^\top,\mathcal{B}_\mathrm{b}^\top,0]}(Q)\leq 0$ have the block form $P = \operatorname{diag}(I_{m+\ell}, P_2, I_{m+\ell})$ and $Q = \operatorname{diag}(I_{m+\ell}, Q_2, I_{m+\ell})$ for some $P_2, Q_2 \in \mathbb{R}^{(p_1+p_2)\times(p_1+p_2)}$. This in particular holds for the minimal solutions of the two KYP inequalities, which we therefore write as $\operatorname{diag}(I_{m+\ell}, \Pi, I_{m+\ell})$ for some suitable diagonal matrix $\Pi \in \mathbb{R}^{(p_1+p_2)\times(p_1+p_2)}$.

Straightforward calculations show that the matrix Π solves $\mathscr{W}_{[\mathcal{A}_z,\mathcal{B}_z,\mathcal{C}_z,\mathcal{D}_z]}(\widehat{P}) \leq 0$ and $\mathscr{W}_{[\mathcal{A}_z^\top,\mathcal{C}_z^\top,\mathcal{B}_z^\top,\mathcal{D}_z^\top]}(\widehat{Q}) \leq 0$. Now, the minimality of Π follows directly from the minimality of $\operatorname{diag}(I_{m+\ell}, \Pi, I_{m+\ell})$.
Recall that the zeros of $[\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n]$ with negative real part are given by the union of the eigenvalues of $\mathcal{A}_{44}, \mathcal{A}_{55}$ and $\left[\begin{smallmatrix} 0 & \mathcal{A}_{36} \\ -\mathcal{A}_{36}^\top & \mathcal{A}_{66}\end{smallmatrix}\right]$ from (5.4). As positive real balanced realizations are minimal, the uncontrollable and unobservable modes of $[\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n]$ are removed by the above construction of $[\mathcal{A}_b, \mathcal{B}_b, \mathcal{C}_b]$, while the remaining ones are preserved. The latter are, however, the eigenvalues of $\mathcal{A}_z$, which leads to the fact that the spectrum of $\mathcal{A}_z$ is contained in the open left complex half plane. ∎

The following remark is devoted to positive real balanced truncation of systems $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ structured as in (3.1) for some $G, D \in \mathbb{R}^{n\times n}$ with $D = D^\top \geq 0$, $G \in \mathrm{Gl}_n(\mathbb{R})$ and $B \in \mathbb{R}^{n\times m}$ by using the approach as in (3.2)–(3.7).

Remark 5.5. *Let a stabilizable system $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ be given. Assume that $\mathcal{A} \in \mathbb{R}^{2n\times 2n}$ and $\mathcal{B}, \mathcal{C}^\top \in \mathbb{R}^{2n\times m}$ are structured as in (3.1) for some $G, D \in \mathbb{R}^{n\times n}$ with $D = D^\top \geq 0$, $G \in \mathrm{Gl}_n(\mathbb{R})$ and $B \in \mathbb{R}^{n\times m}$. Suppose that the system has semi-simple zeros. We apply positive real balanced truncation by using the approach as in (3.2)–(3.7). In particular, we assume that there exist $r^+, r^- \in \mathbb{N}$ that fulfill (3.2)–(3.4) and $r^+ = r^- =: r$ such that the positive real characteristic values $\sigma^\pm_{q^\pm+1}, \ldots, \sigma^\pm_{h^\pm}$ are all strictly below one. Note that Theorem 5.3 implies that $r \geq \ell + m$, where 2ℓ is the number of nonzero and purely imaginary zeros of $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ (counted with multiplicities).*

A positive real realization of the reduced order system can be constructed directly from the realization $[\mathcal{A}_b, \mathcal{B}_b, \mathcal{C}_b]$ in Theorem 5.4 by truncating certain rows and columns which are belonging to the third and fourth block rows of the matrices with block structure as in (5.7). More precisely, the reduced order model has a realization of the form

$$[T^\top \mathcal{A}_b T, T^\top \mathcal{B}_b, \mathcal{C}_b T] \quad \text{for } T = \operatorname{diag}\left(\begin{bmatrix} I_r \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I_r \end{bmatrix}\right). \tag{5.9}$$

However, a direct application of Theorem 3.2 does not necessarily result into the system $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}] = [T^\top \mathcal{A}_b T, T^\top \mathcal{B}_b, \mathcal{C}_b T]$, but rather in a system which is similar to $[T^\top \mathcal{A}_b T, T^\top \mathcal{B}_b, \mathcal{C}_b T]$ under a state space transformation which preserves the property of the signature structure together with $\operatorname{diag}(\Sigma_1^-, \Sigma_1^+)$ being a solution of the KYP inequalities for $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ and $[\widetilde{\mathcal{A}}^\top, \widetilde{\mathcal{C}}^\top, \widetilde{\mathcal{B}}^\top]$. Such transformations are of block-diagonal and orthogonal type, where the sizes of the orthogonal block correspond to the multiplicities of the respective positive real characteristic values. As a consequence, the reduced system can be represented by

$$[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}] = [\widetilde{T}^\top \mathcal{A}_b \widetilde{T}, \widetilde{T}^\top \mathcal{B}_b, \mathcal{C}_b \widetilde{T}],$$

where for some orthogonal matrices $\widetilde{U}_j^\pm \in \mathbb{R}^{n_j^\pm \times n_j^\pm}$ for $j = 1, \ldots, h^\pm$

$$\widetilde{T} = \operatorname{diag}(\widetilde{U}_1^-, \ldots \widetilde{U}_{h^-}^-, \widetilde{U}_{h^+}^+, \ldots, \widetilde{U}_1^+) \cdot \operatorname{diag}\left(\begin{bmatrix} I_r \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I_r \end{bmatrix}\right).$$

6. Construction of second order realizations. Now, we treat the second order realization problem. In particular, we prove a necessary condition on the zero sign characteristics of the reduced system for having a representation as a second order realization. An important tool hereby are the so called *standard triples* [14–16, 23].

Definition 6.1. *A standard triple is defined as a triple* $(X, Z, Y) \in \mathbb{R}^{n\times 2n} \times \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{2n\times n}$ *such that*

$$\begin{bmatrix} X \\ XZ \end{bmatrix} \in \mathrm{Gl}_{2n}(\mathbb{R}) \quad \text{and} \quad \begin{bmatrix} X \\ XZ \end{bmatrix} Y = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}$$

for some nonsingular matrix $M \in \mathbb{R}^{n\times n}$.

Let $\mathbf{L}(s) = s^2M + sD + K$ *for some* $M \in \mathrm{Gl}_n(\mathbb{R})$ *and* $D, K \in \mathbb{R}^{n\times n}$. *Then* $\mathbf{L}(s)$ *is said to be* generated *by a standard triple* (X, Z, Y) *if* $M = (XZY)^{-1}$ *and the equation*

$$MXZ^2 + DXZ + KX = 0$$

is fulfilled.

The tuple (X, Z) from a standard triple together with a mass matrix M completely determine a quadratic matrix polynomial with nonsingular leading term as the next result shows.

Theorem 6.2. [21, Thm. 1] *A standard triple* $(X, Z, Y) \in \mathbb{R}^{n\times 2n} \times \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{2n\times n}$ *generates a uniquely defined quadratic matrix polynomial.*

Actually, in [21] standard triples are considered, where Z is in Jordan canonical form, but it is not made use of this additional property in the proof of the theorem above. Even though a quadratic matrix polynomial is uniquely defined by a standard triple, the reverse is not true in general. This can be seen as there is a one-to-one correpondence between standard triples of some quadratic matrix polynomial $\mathbf{L}(s)$ and realizations of $\mathbf{L}^{-1}(s)$.

Theorem 6.3. [22, Thm. 14.2] *Let* $\mathbf{L}(s) = s^2M + sD + K$ *for some* $M \in \mathrm{Gl}_n(\mathbb{R})$ *and* $D, K \in \mathbb{R}^{n\times n}$. *Then,* $(X, Z, Y) \in \mathbb{R}^{n\times 2n} \times \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{2n\times n}$ *is a standard triple of* $\mathbf{L}(s)$*, if and only if* $\mathbf{L}(s)^{-1} = X(sI_{2n} - Z)^{-1}Y$. *Moreover,* $[Z, Y, X]$ *is minimal as a realization of* $\mathbf{L}^{-1}(s)$. *In particular, if* K^{-1} *exists, then* $K^{-1} = -XZ^{-1}Y$.

It can be further inferred from Theorem 6.2 that the coefficients of $\mathbf{L}(s)$ are given by the so-called *moments* $\Gamma_j := XZ^jY$, i. e.,

$$M = \Gamma_1^{-1}, \quad D = -M\Gamma_2M, \quad K = -M\Gamma_3M + D\Gamma_1D. \tag{6.1}$$

A standard triple (X, Z, Y) is said to be *self-adjoint* if there exists a symmetric matrix $S \in \mathrm{Gl}_n(\mathbb{R})$, such that $SY = X^\top$, $Z^\top = SZS^{-1}$. The theorem above states that $[Z, Y, X]$ is the realization of a symmetric transfer function und thus S is unique. Note that if $\mathbf{L}(s)$ possesses a self-adjoint standard triple, then the moments Γ_j are symmetric and hence, by (6.1) the coefficients of $\mathbf{L}(s)$ are symmetric as well. Thus, by Theorem 6.3 together with (4.4), if one standard triple is self-adjoint, then all standard triples of $\mathbf{L}(s)$ are self-adjoint. Moreover, in this case the pole sign characteristics of $\mathbf{L}^{-1}(s)$ are given by the sign characteristics of (S, Z) for any self-adjoint standard triple $(X, Z, S^{-1}X^\top)$ of $\mathbf{L}(s)$. We need to introduce some further notation.

Definition 6.4. *Let* $(S, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ *be given, where* $S \in \mathrm{Gl}_n(\mathbb{R})$ *is symmetric and* A *is* S*-self-adjoint and diagonalizable over* $\mathbb{C}$. *For* $\alpha < 0$*, we denote by*

$n_-(\alpha)$ $(p_-(\alpha))$ *the number of eigenvalues of* (S, A) *of negative (positive) type in* $[\alpha, 0)$ $((\alpha, 0])$ *and for* $\alpha > 0$, $n_+(\alpha)$ $(p_+(\alpha))$ *the number of eigenvalues of* (S, A) *of negative (positive) type in* $[0, \alpha)$ $((0, \alpha])$.

Next we present a result from [24] that gives a necessary condition on the sign characteristics of quadratic matrix polynomials with positive definite leading and trailing coefficient.

THEOREM 6.5. [24, Thm. 16] *Let* $(S, A) \in \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{2n\times 2n}$ *be given, where* $S = U^\top \mathscr{S}_n U$ *for some* $U \in \mathrm{Gl}_{2n}(\mathbb{R})$ *and* A *is* S*-self-adjoint and diagonalizable over* $\mathbb{C}$. *Let* $\lambda_{\min}$ *and* $\lambda_{\max}$ *be its minimal and maximal real eigenvalue, respectively. Then there exists* $X \in \mathbb{R}^{n\times 2n}$ *such that* $X(sI_{2n} - A)^{-1}S^{-1}X^\top = (s^2M + sD + K)^{-1}$ *for some* $M > 0$, $K \geq 0$ *and* $D = D^\top$, *if and only if*

$$\begin{aligned} n_-(\alpha) &= p_-(\alpha) \quad \forall\, \alpha < \lambda_{\min} \quad \textit{and} \quad n_-(\alpha) \leq p_-(\alpha) \quad \forall\, \alpha \in [\lambda_{\min}, 0), \\ n_+(\alpha) &= p_+(\alpha) \quad \forall\, \alpha > \lambda_{\max} \quad \textit{and} \quad n_+(\alpha) \geq p_+(\alpha) \quad \forall\, \alpha \in (0, \lambda_{\max}]. \end{aligned} \tag{6.2}$$

Remark 6.6. *In [24, Thm. 16], pairs* (S, A) *are considered with a structure similar to the canonical form of Theorem 4.2. However, we have seen earlier that the sign characteristics of* (S, A) *from any self-adjoint standard triple* $(X, A, S^{-1}X^\top)$ *of* $\mathbf{L}(s)$ *are exactly the pole sign characteristics of* $\mathbf{L}^{-1}(s)$ *and hence, do not depend on the choice of the minimal realization of* $\mathbf{L}^{-1}(s)$ *or the standard triple of* $\mathbf{L}(s)$.

We are now able to prove the main result of this section on a necessary and sufficient condition for our reduced system possessing a second order realization with positive definite leading and trailing coefficient. Note that the proof of this theorem is constructive, i. e., we can infer a method for constructing such a second order realization. We restrict ourselves to the case where the reduced system is minimal. However, later in Section 8, when we present the whole reduction procedure, minimality is not required.

THEOREM 6.7. *Let a stabilizable system* $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ *be given in which* $\mathcal{A} \in \mathbb{R}^{2n\times 2n}$ *and* $\mathcal{B} \in \mathbb{R}^{2n\times m}$ *are structured as in (3.1) for some* $G, D \in \mathbb{R}^{n\times n}$ *with* $D = D^\top \geq 0$, $G \in \mathrm{Gl}_n(\mathbb{R})$ *and* $B \in \mathbb{R}^{n\times m}$. *Suppose that the system has semi-simple zeros. Let further* $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ *be the reduced system from Theorem 3.2 of order* $2r$, *where* $r^+ = r^- =: r$. *Suppose it is minimal and its zeros are semi-simple. Then, the following are equivalent:*

- *i) The transfer function* $\widetilde{\mathbf{G}}(s)$ *of* $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ *has a second order realization of the form (1.2), where* $\widetilde{K}, \widetilde{M} > 0$ *and* $\widetilde{D} = \widetilde{D}^\top \in \mathbb{R}^{r\times r}$ *has at least* $r - m$ *positive eigenvalues.*
- *ii) The real zeros of* $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ *are given by* $\mu_1^- \leq \ldots \leq \mu_k^-$ *and* $\mu_1^+ \leq \ldots \leq \mu_k^+ < \mu_{k+1}^+ = \ldots = \mu_{k+m}^+ = 0$ *and fulfill* $\mu_i^- < \mu_i^+$ *for all* $i = 1, \ldots, k$.
- *iii) The transfer function* $\widetilde{\mathbf{G}}(s)$ *has a minimal realization* $[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}]$ *structured as in (5.4) that fulfills the properties a)–c) from Theorem 5.3 and* $\mathcal{A}_{44} - \mathcal{A}_{55} = \mathrm{diag}(\mu_1^+, \ldots, \mu_k^+) - \mathrm{diag}(\mu_1^-, \ldots, \mu_k^-) > 0$.

Proof. *ii)* ⇒ *iii)*: Let $\mathbf{G}(s)$ be the transfer function of $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ and $[\widetilde{\mathcal{A}}_\mathrm{b}, \widetilde{\mathcal{B}}_\mathrm{b}, \widetilde{\mathcal{C}}_\mathrm{b}]$ be its the positive real balanced realization given by Theorem 5.4 of the form (5.7). Due to Remark 5.5 we can assume w. l. o. g. that $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}] = [T^\top \mathcal{A}_\mathrm{b} T, T^\top \mathcal{B}_\mathrm{b}, \mathcal{C}_\mathrm{b} T]$ for $T = \mathrm{diag}\left(\left[\begin{smallmatrix} I_r \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ I_r \end{smallmatrix}\right]\right)$. Let

$$[\widetilde{\mathcal{A}}_\mathrm{z}, \widetilde{\mathcal{B}}_\mathrm{z}, \widetilde{\mathcal{C}}_\mathrm{z}, \mathcal{D}_\mathrm{z}] = [\widetilde{T}^\top \mathcal{A}_\mathrm{z} \widetilde{T}, \widetilde{T}^\top \mathcal{B}_\mathrm{z}, \mathcal{C}_\mathrm{z} \widetilde{T}, \mathcal{D}_\mathrm{z}]$$

for $\widetilde{T} := \operatorname{diag}\left(\left[\begin{smallmatrix} 0 \\ I_{r-m-\ell} \end{smallmatrix}\right], \left[\begin{smallmatrix} I_{r-m-\ell} \\ 0 \end{smallmatrix}\right]\right)$ and $[\mathcal{A}_{\mathrm{z}}, \mathcal{B}_{\mathrm{z}}, \mathcal{C}_{\mathrm{z}}, \mathcal{D}_{\mathrm{z}}]$ be as in (5.8). Moreover, the eigenvalues of $\widetilde{\mathcal{A}}_{\mathrm{z}}$ are zeros of $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ and the real nonzero zeros of $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ are eigenvalues of $\widetilde{\mathcal{A}}_{\mathrm{z}}$. Since $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ is received by positive real balanced truncation of the minimal and asymptotically stable system $[\mathcal{A}_{\mathrm{z}}, \mathcal{B}_{\mathrm{z}}, \mathcal{C}_{\mathrm{z}}, \mathcal{D}_{\mathrm{z}}]$ from (5.8), the discussion in Section III of [18] provides that the system $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ is asymptotically stable as well. Hence we can apply Corollary 4.3 to find some $\widehat{T} \in \mathrm{Gl}_{2p}(\mathbb{R})$ such that

$$\widehat{T}^{-1}\widetilde{\mathcal{A}}_{\mathrm{z}}\widehat{T} = \begin{bmatrix} 0 & 0 & \mathcal{V} \\ 0 & \Lambda & 0 \\ -\mathcal{V} & 0 & \mathcal{E} \end{bmatrix} \quad \text{and} \quad \widehat{T}^{\top}\mathscr{S}_p\widehat{T} = \mathscr{S}_p,$$

where $\Lambda = \operatorname{diag}(\mu_1^+, \ldots, \mu_k^+, \mu_1^-, \ldots, \mu_k^-)$ and $\mathcal{E} = \operatorname{diag}(\eta_1, \ldots, \eta_c)$ for $c := p - k$, $\mathcal{V} = \operatorname{diag}(\nu_1, \ldots, \nu_c)$, with $\eta_i < 0$ and $\nu_i \in \mathbb{R}$ for $i = 1, \ldots, c$. Applying the state space transformation $\breve{T} := \operatorname{diag}(I_{m+\ell}, \widehat{T}, I_{m+\ell})$ to $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ then gives a minimal realization $[\mathcal{A}_{\mathrm{n}}, \mathcal{B}_{\mathrm{n}}, \mathcal{C}_{\mathrm{n}}]$ of $\widetilde{\mathbf{G}}(s)$ structured as in Theorem 5.3 and which fulfills the properties a)–c) of this theorem. Now the zero sign characteristics of $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ are given by the sign characteristics of $(\operatorname{diag}(I_m, -\mathscr{S}_{2k}), \operatorname{diag}(0, \Lambda))$. Hence, by assumption we get that $\operatorname{diag}(\mu_1^+, \ldots, \mu_k^+) - \operatorname{diag}(\mu_1^-, \ldots, \mu_k^-) > 0$.

iii) $\Rightarrow$ *ii)*: This follows since $\mu_1^+, \ldots, \mu_k^+$ and $\mu_1^-, \ldots, \mu_k^-$ are respectively the negative zeros of positive and negative type of the minimal realization $[\mathcal{A}_{\mathrm{n}}, \mathcal{B}_{\mathrm{n}}, \mathcal{C}_{\mathrm{n}}]$ of $\widetilde{\mathbf{G}}(s)$ which is structured as in (5.4) and $\widetilde{\mathbf{G}}(0) = \mathcal{C}_{\mathrm{n}}\mathcal{A}_{\mathrm{n}}^{-1}\mathcal{B}_{\mathrm{n}} = 0$.

iii) $\Rightarrow$ *i)*: Suppose we are given a minimal realization $[\mathcal{A}_{\mathrm{n}}, \mathcal{B}_{\mathrm{n}}, \mathcal{C}_{\mathrm{n}}]$ structured as in Theorem 5.3 and such that $\operatorname{diag}(\mu_1^+, \ldots, \mu_k^+) - \operatorname{diag}(\mu_1^-, \ldots, \mu_k^-) > 0$. For $i = 1, \ldots, k$ we set

$$a_i := \sqrt{\tfrac{\mu_i^-}{\mu_i^- - \mu_i^+}} \in \mathbb{R}\backslash\{0\}, \quad b_i := \sqrt{\tfrac{\mu_i^+}{\mu_i^- - \mu_i^+}} \in \mathbb{R}\backslash\{0\}, \quad T_i = \begin{bmatrix} a_i & b_i \\ b_i & a_i \end{bmatrix} \in \mathrm{Gl}_2(\mathbb{R}).$$

Straightforward calculations give $T_i^{-1} = \mathscr{S}_2 T_i \mathscr{S}_2$ and

$$T_i^{-1} \operatorname{diag}(\mu_i^+, \mu_i^-) T_i = \begin{bmatrix} 0 & * \\ * & \frac{(\mu_i^-)^2 - (\mu_i^+)^2}{\mu_i^- - \mu_i^+} \end{bmatrix}. \tag{6.3}$$

Note that since $\mu_i^\pm < 0$ for all $i = 1, \ldots, k$, the lower right entry of the matrix on the right-hand side is negative. We set $\widetilde{T} := \left[\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right]$, where $A := \operatorname{diag}(a_1, \ldots, a_k)$ and $B := \operatorname{diag}(b_1, \ldots, b_k)$ and $T := \operatorname{diag}(I_{r-k}, \widetilde{T}, I_{r-k})$. Interchanging some rows and columns, we arrive at a minimal realization $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ of $\widetilde{\mathbf{G}}(s)$ of the form (1.5), where a second order realization can be derived directly with $\widetilde{M} = I_r$, $\widetilde{K} = \widetilde{G}\widetilde{G}^{\top}$ and $\widetilde{D} = \widetilde{D}^{\top}$. Note that $\widetilde{D}$ has the block form $\widetilde{D} = \left[\begin{smallmatrix} \widetilde{D}_{11} & \widetilde{D}_{12} \\ \widetilde{D}_{12}^{\top} & \widetilde{D}_{22} \end{smallmatrix}\right]$, where $\widetilde{D}_{11} \in \mathbb{R}^{r-m \times r-m}$ and $\widetilde{D}_{22} \in \mathbb{R}^{m \times m}$ are positive semidefinite and diagonal matrices. Hence, Sylvester's law of inertia [20, Thm. 4.5.8] implies that $\widetilde{D}$ has at least $r - m$ non-negative eigenvalues.

i) $\Rightarrow$ *iii)*: Suppose $\widetilde{\mathbf{G}}(s)$ has a minimal second order realization of the form (1.2), where $\widetilde{M} \in \mathbb{R}^{r \times r}$. By Lemma 5.1, $\widetilde{\mathbf{G}}(s)$ then also has a minimal realization $[\mathcal{A}_{\mathrm{s}}, \mathcal{B}_{\mathrm{s}}, \mathcal{C}_{\mathrm{s}}]$ of the form (5.1), where its real and nonzero zeros coincide with the real eigenvalues of $\widehat{\mathcal{A}}_{\mathrm{s}}$ and the zero sign characteristics of $[\mathcal{A}_{\mathrm{s}}, \mathcal{B}_{\mathrm{s}}, \mathcal{C}_{\mathrm{s}}]$ coincide with the sign characteristics of $(\operatorname{diag}(I_m, -\mathscr{S}_{r-m}), \operatorname{diag}(0, \widehat{\mathcal{A}}_{\mathrm{s}}))$. Since, as mentioned in the first part, all zeros of $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ have nonpositive real part, we can apply Theorem 5.3 to obtain a minimal realization $[\mathcal{A}_{\mathrm{n}}, \mathcal{B}_{\mathrm{n}}, \mathcal{C}_{\mathrm{n}}]$ of $\widetilde{\mathbf{G}}(s)$ of the form (5.4). Recall that the transformation

$T \in \mathrm{Gl}_{2r}(\mathbb{R})$ we use to transform $[\mathcal{A}_\mathrm{s}, \mathcal{B}_\mathrm{s}, \mathcal{C}_\mathrm{s}]$ into $[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}]$ has the block structure $T = \mathrm{diag}(I_m, \widehat{T}, I_m)$, where $\widehat{T} \in \mathrm{Gl}_{2(r-m)}(\mathbb{R})$ has to fulfill (4.5), i.e., $\widehat{T}^\top \mathscr{S}_{r-m} \widehat{T} = \mathscr{S}_{r-m}$. This implies that the sign characteristics of $(-\mathscr{S}_{r-m}, \widehat{\mathcal{A}}_\mathrm{s})$ coincide with the sign characteristics of $(-\mathscr{S}_k, \mathrm{diag}(\mathcal{A}_{44}, \mathcal{A}_{55}))$, with $\mathcal{A}_{44}, \mathcal{A}_{55}$ from (5.4). Setting $X := \begin{bmatrix} 0 & I_{r-m} \end{bmatrix}$, with $\widehat{\mathcal{A}}_\mathrm{s} = \begin{bmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11} \end{bmatrix}$ as in Lemma 5.1 we obtain

$$\begin{bmatrix} X\widehat{\mathcal{A}}_\mathrm{s}^{-1} \\ X\widehat{\mathcal{A}}_\mathrm{s}^{-1}\widehat{\mathcal{A}}_\mathrm{s} \end{bmatrix} = \begin{bmatrix} -G_{13}^{-\top} & 0 \\ 0 & I_{r-m} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X\widehat{\mathcal{A}}_\mathrm{s}^{-1} \\ X\widehat{\mathcal{A}}_\mathrm{s}^{-1}\widehat{\mathcal{A}}_\mathrm{s} \end{bmatrix} \mathscr{S}_{r-m} X^\top = \begin{bmatrix} 0 \\ I_{m-r} \end{bmatrix}.$$

Thus, using (6.1) and Theorem 6.3, $(X\widehat{\mathcal{A}}_\mathrm{s}^{-1}, \widehat{\mathcal{A}}_\mathrm{s}, \mathscr{S}_{r-m}X^\top)$ is a standard triple for

$$\mu^2 I_{r-m} - \mu X \widehat{\mathcal{A}}_\mathrm{s} \mathscr{S}_{r-m} X^\top - (X(\widehat{\mathcal{A}}_\mathrm{s}^{-1})^2 \mathscr{S}_{r-m} X^\top)^{-1} = \mu^2 I_{r-m} + \mu D_{11} + G_{13} G_{13}^\top.$$

Moreover, the standard triple is S-self-adjoint for $S := \mathscr{S}_{r-m}\widehat{\mathcal{A}}_\mathrm{s}^{-1}$, since

$$\begin{aligned} S\widehat{\mathcal{A}}_\mathrm{s} S^{-1} &= \mathscr{S}_{r-m}\widehat{\mathcal{A}}_\mathrm{s}^{-1}\widehat{\mathcal{A}}_\mathrm{s}\widehat{\mathcal{A}}_\mathrm{s}\mathscr{S}_{r-m} = \mathscr{S}_{r-m}\widehat{\mathcal{A}}_\mathrm{s}\mathscr{S}_{r-m} = \widehat{\mathcal{A}}_\mathrm{s}^\top, \\ (X\widehat{\mathcal{A}}_\mathrm{s}^{-1})^\top &= \widehat{\mathcal{A}}_\mathrm{s}^{-\top} X^\top = \widehat{\mathcal{A}}_\mathrm{s}^{-\top} \mathscr{S}_{r-m}\mathscr{S}_{r-m} X^\top = \mathscr{S}_{r-m}\widehat{\mathcal{A}}_\mathrm{s}^{-1}\mathscr{S}_{r-m} X^\top = S\mathscr{S}_{r-m}X^\top. \end{aligned}$$

Now Theorem 6.5 implies that for this standard triple, $n_-(\alpha) \le p_-(\alpha)$ for all $\alpha \in [\mu_{\min}, 0)$. Further, we get that

$$\widehat{T}^\top \widehat{\mathcal{A}}_\mathrm{s}^{-\top} \widehat{T}^{-\top} = \begin{bmatrix} -\mathcal{A}_{36}^{-\top}\mathcal{A}_{66}\mathcal{A}_{36}^{-1} & 0 & 0 & \mathcal{A}_{36}^{-\top} \\ 0 & \mathcal{A}_{44}^{-1} & 0 & 0 \\ 0 & 0 & \mathcal{A}_{55}^{-1} & 0 \\ -\mathcal{A}_{36}^{-1} & 0 & 0 & 0 \end{bmatrix}.$$

The sign characteristics only depend on the real eigenvalues and thus, $(\mathscr{S}_{r-m}\widehat{\mathcal{A}}_\mathrm{s}^{-1}, \widehat{\mathcal{A}}_\mathrm{s})$ and $(\widehat{T}^\top \mathscr{S}_{r-m}\widehat{T}\widehat{T}^{-1}\widehat{\mathcal{A}}_\mathrm{s}^{-1}\widehat{T}, \widehat{T}^{-1}\widehat{\mathcal{A}}_\mathrm{s}\widehat{T})$ have the same sign characteristics as the tuple $(\mathrm{diag}(\mathcal{A}_{44}^{-1}, \mathcal{A}_{55}^{-1})\mathscr{S}_k, \mathrm{diag}(\mathcal{A}_{44}, \mathcal{A}_{55}))$. Recall that $\mathcal{A}_{44}$ and $\mathcal{A}_{55}$ are negative definite. Hence, the sign characteristics of the latter coincide with the sign characteristics of $(-\mathscr{S}_k, \mathrm{diag}(\mathcal{A}_{44}, \mathcal{A}_{55}))$. Therefore, we get that $n_-(\alpha) \le p_-(\alpha)$ for all $\alpha \in [\mu_{\min}, 0)$ also holds for $(-\mathscr{S}_k, \mathrm{diag}(\mathcal{A}_{44}, \mathcal{A}_{55}))$, which means that $\mathcal{A}_{55} < \mathcal{A}_{44}$. ∎

We close this section with a discussion on the treatment of the necessary condition that is needed in order to recover a second order realization which we have learned previously.

Remark 6.8 (Handling of the necessary condition). *Unfortunately, the condition* $\mathrm{diag}(\mu_1^+, \ldots, \mu_k^+) - \mathrm{diag}(\mu_1^-, \ldots, \mu_k^-)$ *on the zeros of the reduced system is not fulfilled in general. Moreover, we cannot guarantee to establish this property by small perturbations, since the sign characteristics are stable in the following sense. Consider a pair* $(H, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$*, where* H *is symmetric and nonsinuglar and* A *is* H*-self-adjoint and diagonalizable over* $\mathbb{C}$*. Then for every simple real eigenvalue* λ*, there exist structure preserving neighborhoods* U_A *of* A*,* U_H *of* H *and* U_λ *of* λ*, such that if* $A_1 \in U_A$ *is* H_1*-self-adjoint for some* $H_1 \in U_H$*, then there is exactly one real simple eigenvalue* λ_1 *of* A_1 *in* U_λ *and it holds that it has the same sign as* λ*, see [15, Sec. III.5.1, Thm. 5.1]. Due to these facts, we have decided that in our reduction method to add blocks instead of relying on perturbations of the system matrices, whenever it is necessary, see Section 8.*

7. Positive real balanced truncation for overdamped systems - the exceptional case. Here we consider second order systems of the form (1.1) that additionally fulfill that M, D and K are all symmetric and positive definite, and the so called *overdamping condition*

$$(v^*Dv)^2 > 4(v^*Mv)(v^*Kv) \text{ for all } v \in \mathbb{C}^n \tag{7.1}$$

is fulfilled. It is known that the overdamping condition implies that all zeros of $\det(s^2M + sD + K)$ are real, see [9]; it therefore generalizes the discriminant condition to the matrix-valued case. By forming a self-adjoint standard triple $(X, A, S^{-1}X^\top)$ of $\mathbf{L}(s) = s^2M + sD + K$, we can define the sign characteristics of $\mathbf{L}(s)$ to be those of (S, A) [13]. The sign characteristics of overdamped systems have a certain structure, which is summarized in the following result.

Theorem 7.1. *[1, Thm 3.6] Let $(X, A, S^{-1}X^\top)$ form a self-adjoint standard triple of some matrix polynomial $s^2M + sD + K$, where $M, D, K \in \mathbb{R}^{n\times n}$ are symmetric. Then the following are equivalent:*

i) All eigenvalues of (S, A) are real and negative and every eigenvalue of negative type is smaller than every eigenvalue of positive type. In detail, they fulfill $\lambda_1^- \leq \ldots \leq \lambda_n^- < \lambda_1^+ \leq \ldots \leq \lambda_n^+ < 0$.

ii) It holds that $M, D, K > 0$ and (7.1).

Noticing that the distribution of the real eigenvalues and their signs are system invariants, we obtain the following result.

Corollary 7.2. *Let a second order system be given of the form (1.1), with $M, D, K > 0$, which fulfills the condition (7.1) and let $\mathbf{G}(s)$ be its transfer function. Then every second order realization of the form (1.1) with symmetric $\widetilde{M}, \widetilde{D}, \widetilde{K} \in \mathbb{R}^{n\times n}$ fulfills $\widetilde{M}, \widetilde{D}, \widetilde{K} > 0$ and (7.1). In this case, for the first order representation $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ of the form (1.3), the eigenvalues of $(\mathscr{S}_n, \mathcal{A})$ fulfill $\lambda_1^- \leq \ldots \leq \lambda_n^- < \lambda_1^+ \leq \ldots \leq \lambda_n^+ < 0$.*

In order to show that positive real balanced truncation preserves the overdamping structure, we use some results of [6] and therefore need some notation.

Definition 7.3. *Let $(S, A) \in \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{2n\times 2n}$, where S is symmetric and nonsingular, A is S-self-adjoint and diagonalizable over $\mathbb{C}$. Consider the set $C^{\pm} := \{x \in \mathbb{C}^{2n} \mid x^*Sx \gtrless 0\}$ and, for a subspace $\mathcal{S} \subseteq \mathbb{C}^{2n}$, let*

$$\iota^+(\mathcal{S}) := \begin{cases} +\infty & \text{if } \mathcal{S} \cap C^+ = \emptyset, \\ \sup\limits_{x\in\mathcal{S}\cap C^+} \rho(x) & \text{otherwise} \end{cases}, \quad \iota^-(\mathcal{S}) := \begin{cases} -\infty & \text{if } \mathcal{S} \cap C^- = \emptyset, \\ \inf\limits_{x\in\mathcal{S}\cap C^-} \rho(x) & \text{otherwise,} \end{cases} \tag{7.2}$$

where $\rho(x) := \frac{x^*SAx}{x^*Sx}$. We denote by

$$\begin{aligned} \delta_h^+(S, A) &:= \{\inf \iota^+(\mathcal{S}) \mid \mathcal{S} \text{ subspace of } \mathbb{C}^{2n} \text{ with } \dim \mathcal{S} = 2n - h + 1\} \\ \text{and} \quad \sigma_h^-(S, A) &:= \{\sup \iota^-(\mathcal{S}) \mid \mathcal{S} \text{ subspace of } \mathbb{C}^{2n} \text{ with } \dim \mathcal{S} = 2n - h + 1\}. \end{aligned}$$

Note that one has $\delta_{2n}^+(S, A) \leq \ldots \leq \delta_1^+(S, A)$ and $\sigma_1^-(S, A) \leq \ldots \leq \sigma_{2n}^-(S, A)$. We present two results from [6]. The second one treats the assignment of the values $\sigma_j^-(S, A)$ and $\delta_j^+(S, A)$ to eigenvalues of positive and negative type for tuples (S, A) as in Definition 7.3 where all eigenvalues are real and $\lambda_1^- \leq \ldots \leq \lambda_n^- \leq \lambda_1^+ \leq \ldots \leq \lambda_n^+$. In the first result, we study the values $\sigma_j^-(S, A)$ for tuples (S, A) that do not satisfy such a distribution of the sign characteristics.

Lemma 7.4. *[6, Lem. 3.2] Let* $(S,A) \in \mathbb{R}^{2n\times 2n}\times\mathbb{R}^{2n\times 2n}$, *where* $S = U^\top \mathscr{S}_n U$ *for some* $U \in \mathrm{Gl}_{2n}(\mathbb{R})$ *and* A *is* S*-self-adjoint and diagonalizable over* $\mathbb{C}$. *Suppose the eigenvalues of* (S,A) *do not satisfy* $\lambda_1^- \leq \ldots \leq \lambda_n^- \leq \lambda_1^+ \leq \ldots \leq \lambda_n^+$, *i. e.,* (S,A) *has at least one non-real eigenvalue, or there exists an eigenvalue of positive type which is larger than an eigenvalue of negative type. Then,* $\sigma_{n+j}^-(S,A) = \infty$ *for* $1 \leq j \leq n$.

Theorem 7.5. *[6, Cor. 4.4 & Thm. 4.3] Let* $(S,A) \in \mathbb{R}^{2n\times 2n}\times\mathbb{R}^{2n\times 2n}$, *where* $S = U^\top \mathscr{S}_n U$ *for some* $U \in \mathrm{Gl}_{2n}(\mathbb{R})$ *and* A *is* S*-self-adjoint and diagonalizable. Suppose that all eigenvalues of* (S,A) *are real and satisfy* $\lambda_1^- \leq \ldots \leq \lambda_n^- \leq \lambda_1^+ \leq \ldots \leq \lambda_n^+$. *Then, for* $1 \leq j \leq n$ *we have*

$$\begin{aligned} \sigma_j^-(S,A) &= -\infty, & & & \sigma_{n+j}^-(S,A) &= \lambda_j^-, \\ \delta_{2n-j+1}^+(S,A) &= \lambda_j^+, & & \text{and} & \delta_j^+(S,A) &= \infty. \end{aligned}$$

In particular, if $\lambda_n^- = \lambda_1^+$, *then* $\sigma_{2n}^-(S,A) = \lambda_n^-(S,A) = \lambda_1^+(S,A) = \delta_{2n}^+(S,A)$.

A consequence of Theorem 7.5 is that for S-self-adjoint and diagonalizable A with only real eigenvalues which additionally fulfill $\lambda_1^- \leq \ldots \leq \lambda_n^- \leq \lambda_1^+ \leq \ldots \leq \lambda_n^+$, this property is preserved under the application of certain two-sided reduction matrices on the pair (S,A).

Lemma 7.6. *Let* $(S,A) \in \mathbb{R}^{2n\times 2n}\times\mathbb{R}^{2n\times 2n}$, *where* $S = U^\top \mathscr{S}_n U$ *for some* $U \in \mathrm{Gl}_{2n}(\mathbb{R})$ *and* A *is* S*-self-adjoint. Suppose that* A *is diagonalizable over* $\mathbb{C}$ *and that* (S,A) *has only real eigenvalues that satisfy* $\lambda_1^- \leq \ldots \leq \lambda_n^- < \lambda_1^+ \leq \ldots \leq \lambda_n^+$. *Let* $W, V \in \mathbb{R}^{2n\times r}$ *be full rank matrices with* $r \leq 2n$ *and suppose that for a symmetric matrix* $H \in \mathbb{R}^{r\times r}$ *it holds that* $HW^\top = V^\top S$ *and* $W^\top V = I_r$. *If the eigenvalues of* $(H, W^\top AV)$ *are semi-simple, then they are all real and, if we denote them by* $\widetilde{\lambda}_1^\pm \leq \ldots \leq \widetilde{\lambda}_{r^\pm}^\pm$, *where* $r^+ + r^- = r$, *they fulfill* $\widetilde{\lambda}_1^- \leq \ldots \leq \widetilde{\lambda}_{r^-}^- < \widetilde{\lambda}_1^+ \leq \ldots \leq \widetilde{\lambda}_{r^+}^+$.

Proof. We set $B := W^\top AV$ and for $v \in \mathbb{C}^r$ we write

$$\widetilde{\rho}(v) := \frac{v^* V^\top SAVv}{v^* V^\top SVv} = \frac{v^* HBv}{v^* Hv}.$$

By the previous Theorem we have that $\lambda_n^- = \sigma_{2n}^-(S,A) = \sup\{\rho(x) \mid x \in C^-\}$. Then,

$$\begin{aligned} \sigma_r^-(H,B) = \sup\{\widetilde{\rho}(v) \mid v \in \mathbb{C}^r,\ v^*Hv > 0\} &= \sup\{\rho(x) \mid x \in \mathbb{C}^{2n},\, x \in C^- \cap \operatorname{im} V\} \\ &\leq \sigma_{2n}^-(S,A) < \infty. \end{aligned}$$

Now, from Lemma 7.4 it follows that all eigenvalues of (H,B) are real and $\widetilde{\lambda}_1^- \leq \ldots \leq \widetilde{\lambda}_{r^-}^- \leq \widetilde{\lambda}_1^+ \leq \ldots \leq \widetilde{\lambda}_{r^+}^+$. Hence, it remains to show that $\widetilde{\lambda}_{r^-}^- < \widetilde{\lambda}_1^+$. Note that

$$\begin{aligned} \delta_r^+(H,B) = \inf\{\widetilde{\rho}(v) \mid v \in \mathbb{C}^r,\ v^*Hv < 0\} &= \inf\{\rho(x) \mid x \in \mathbb{C}^{2n},\, x \in C^+ \cap \operatorname{im} V\} \\ &\geq \inf\{\rho(x) \mid x \in C^+\} = \delta_{2n}^+(S,A). \end{aligned}$$

In particular, by Theorem 7.5 it holds that $\widetilde{\lambda}_{r^-}^- = \widetilde{\sigma}_r^-(H,B) \leq \sigma_{2n}^-(S,A) = \lambda_n^- < \lambda_1^- = \delta_{2n}^+(S,A) \leq \delta_r^+(H,B) = \widetilde{\lambda}_1^+$. ∎

For the main theorem of this section we combine the lemma above with the results from Section 5. More precisely, we prove that the structure of an overdamped second order system is preserved applying the reduction ansatz from Section 3. Naturally, the reduction matrices V and $W^\top$ from (3.7) that are used to derive the reduced system fulfill the requirements of the lemma above, i. e., $W^\top V = I$ and $\mathscr{S}_r W^\top = V^\top \mathscr{S}_n$.

Nevertheless, in order to prove that the poles and zeros of the reduced model are all smaller than zero we have to work a bit more.

THEOREM 7.7. *Let a stabilizable system $[\mathcal{A},\mathcal{B},\mathcal{C}]$ be given. Assume that $\mathcal{A} \in \mathbb{R}^{2n\times 2n}$ and $\mathcal{B} \in \mathbb{R}^{2n\times m}$ are structured as in (3.1) for some $G, D \in \mathbb{R}^{n\times n}$ with $D = D^\top \geq 0$, $G \in \mathrm{Gl}_n(\mathbb{R})$ and $B \in \mathbb{R}^{n\times m}$. Let $M := I_n$ and $K := GG^\top$ and assume that $D > 0$ and they fulfill (7.1). Let further $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}]$ be the reduced system from Theorem 3.2 of order $2r$ for $r := r^+ = r^-$ and let $\widetilde{\mathbf{G}}(s)$ be its transfer function. Suppose that the eigenvalues of $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ and the zeros of $[\mathcal{A},\mathcal{B},\mathcal{C}]$ and $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}]$ are all semi-simple. Then all eigenvalues of $\mathcal{A}$ are real and fulfill $\widetilde{\lambda}_1^- \leq \ldots \leq \widetilde{\lambda}_r^- < \widetilde{\lambda}_1^+ \leq \ldots \leq \widetilde{\lambda}_r^+$. In particular, every second order realization of $\widetilde{\mathbf{G}}(s)$ of the form (1.1) with symmetric $\widetilde{M}, \widetilde{D}, \widetilde{K} \in \mathbb{R}^{r\times r}$ fulfills $\widetilde{M}, \widetilde{D}, \widetilde{K} > 0$ and (7.1). Moreover, $\widetilde{\mathbf{G}}(s)$ possesses such a second order realization.*

Proof. A combination of Lemma 5.1 and Corollary 7.2 implies that we can w.l.o.g. assume that $[\mathcal{A}_\mathrm{s},\mathcal{B}_\mathrm{s},\mathcal{C}_\mathrm{s}]$ is of the form (5.1) and all eigenvalues of $(\mathscr{S}_n,\mathcal{A})$ are real, negative, and they fulfill $\lambda_1^- \leq \ldots \leq \lambda_n^- < \lambda_1^+ \leq \ldots \leq \lambda_n^+ < 0$. Since $D > 0$, the second last block rows and columns of size ℓ in (5.1) are absent. Let $\mathbf{G}(s)$ denote the transfer function of $[\mathcal{A},\mathcal{B},\mathcal{C}]$ and let $[\mathcal{A}_\mathrm{b},\mathcal{B}_\mathrm{b},\mathcal{C}_\mathrm{b}]$ be a positive real balanced realization of the form (5.7) which exists by Theorem 5.4. Due to Remark 5.5 we can w. l. o. g. assume that $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}] = [T^\top\mathcal{A}_\mathrm{b}T, T^\top\mathcal{B}_\mathrm{b}, \mathcal{C}_\mathrm{b}T]$ for $T = \mathrm{diag}\left(\left[\begin{smallmatrix} I_r \\ 0\end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ I_r\end{smallmatrix}\right]\right)$. Let

$$[\widetilde{\mathcal{A}}_\mathrm{z},\widetilde{\mathcal{B}}_\mathrm{z},\widetilde{\mathcal{C}}_\mathrm{z},\mathcal{D}_\mathrm{z}] = [\widetilde{T}^\top\mathcal{A}_\mathrm{z}\widetilde{T}, \widetilde{T}^\top\mathcal{B}_\mathrm{z}, \mathcal{C}_\mathrm{z}\widetilde{T}, \mathcal{D}_\mathrm{z}]$$

for $\widetilde{T} := \mathrm{diag}\left(\left[\begin{smallmatrix} 0 \\ I_{r-m}\end{smallmatrix}\right], \left[\begin{smallmatrix} I_{r-m} \\ 0\end{smallmatrix}\right]\right)$ and $[\mathcal{A}_\mathrm{z},\mathcal{B}_\mathrm{z},\mathcal{C}_\mathrm{z},\mathcal{D}_\mathrm{z}]$ as in (5.8). Since the eigenvalues of $\mathcal{A}_\mathrm{b}$ are a subset of the eigenvalues of $\mathcal{A}$ they are all real and the eigenvalues of negative type of $(\mathrm{diag}(-I_{m+p_1}, I_{p_2+m}), \mathcal{A}_\mathrm{b})$ are all strictly smaller than the eigenvalues of positive type of $(\mathrm{diag}(-I_{m+p_1}, I_{p_2+m}), \mathcal{A}_\mathrm{b})$. Now the tuple $(\mathrm{diag}(-I_{m+p_1}, I_{p_2+m}), \mathcal{A}_\mathrm{b})$ and reduction matrices $V := T$ and $W^\top := T^\top$ fulfill the assumptions of Lemma 7.6 and hence, so does $(T^\top \mathrm{diag}(-I_{m+p_1}, I_{p_2+m})T, T^\top\mathcal{A}_\mathrm{b}T) = (\mathscr{S}_r, \widetilde{\mathcal{A}})$ and $\widetilde{T}$. It follows that the eigenvalues of $(\mathscr{S}_r, \widetilde{\mathcal{A}})$ and of $(\mathscr{S}_{r-m}, \widetilde{\mathcal{A}}_\mathrm{z})$ are all real and fulfill $\widetilde{\lambda}_1^- \leq \ldots \leq \widetilde{\lambda}_r^- < \widetilde{\lambda}_1^+ \leq \ldots \leq \widetilde{\lambda}_r^+$ and $\widetilde{\mu}_1^- \leq \ldots \leq \widetilde{\mu}_{r-m}^- < \widetilde{\mu}_1^+ \leq \ldots \leq \widetilde{\mu}_{r-m}^+$, respectively. On the other hand, the system $[\widetilde{\mathcal{A}}_\mathrm{z},\widetilde{\mathcal{B}}_\mathrm{z},\widetilde{\mathcal{C}}_\mathrm{z},\mathcal{D}_\mathrm{z}]$ is received by positive real balanced truncation of the asymptotically stable and positive real balanced (and thus, minimal) system $[\mathcal{A}_\mathrm{z},\mathcal{B}_\mathrm{z},\mathcal{C}_\mathrm{z},\mathcal{D}_\mathrm{z}]$. Now, the discussion in Section III of [18] provides that the system $[\widetilde{\mathcal{A}}_\mathrm{z},\widetilde{\mathcal{B}}_\mathrm{z},\widetilde{\mathcal{C}}_\mathrm{z},\mathcal{D}_\mathrm{z}]$ is asymptotically stable as well and hence, $\widetilde{\mu}_{r-m}^+ < 0$. In particular, $\widetilde{\mathcal{A}}_\mathrm{z}$ is invertible. Since by Theorem 5.4, $\widehat{\mathcal{A}}_{16} \in \mathrm{Gl}_m(\mathbb{R})$, which is the block matrix on the upper right of $\mathcal{A}_\mathrm{b}$ and also of $\widetilde{\mathcal{A}}$, it follows from the block form of $\widetilde{\mathcal{A}}$ that it is invertible as well. By the passivity of the reduced system we have that $\widetilde{\lambda}_r^+ \leq 0$ and hence, $\widetilde{\lambda}_r^+ < 0$. We have now shown that property a) from Theorem 7.1 is fulfilled for the pole sign characteristics of the reduced system.

Since the second condition of Theorem 6.7 is fulfilled, we follow the steps "*ii*) ⇒ *iii*)" and "*iii*) ⇒ *i*)" to obtain a state space transformation $\breve{T} \in \mathrm{Gl}_{2r}(\mathbb{R})$ such that $\breve{T}^{-1}\widetilde{\mathcal{A}}\breve{T}$, $\breve{T}^{-1}\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{C}}\breve{T}$ are structured as in (3.1) for some matrices $\widetilde{G} \in \mathrm{Gl}_r(\mathbb{R})$, $\widetilde{B} \in \mathbb{R}^{r\times m}$ and some symmetric matrix $\widetilde{D} \in \mathbb{R}^{r\times r}$. Note that the minimality assumption of Theorem 6.7 is not needed in order to derive the transformation $\breve{T}$. This shows that $\widetilde{\mathbf{G}}(s)$ possesses a second order realization of the form (1.1) with symmetric coefficients $\widetilde{M} = I_r$, $\widetilde{K} = \widetilde{G}\widetilde{G}^\top$, and $\widetilde{D}$. Now Theorem 7.1 provides that it fulfills $\widetilde{M}, \widetilde{D}, \widetilde{K} > 0$ and (7.1). The rest of the theorem follows from Corollary 7.2. ∎

8. Numerical aspects. This part is devoted to the numerical issues of the presented results. One problem that occurs considering the original large-scale system is that we need some factorization of the mass and stiffness matrices M and K. Here, for many applications in mechanics those are band matrices or have an equally sparse structure, a property that one can be exploited to compute sparse Cholesky factorizations.

The bottleneck in the computation of the reduced-order system the the determination of the minimal solution of the KYP inequality $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},0]}(P) \leq 0$. Here one can use the method from [28], which provides the minimal solution directly in factored form. That is, this method delivers low rank approximations of the type $P_{\min} \approx L^\top L$ for a matrix L which has a small number of rows compared to the number of columns. We arrive at the following numerical procedure.

Numerical procedure: Second order positive real balanced truncation and structure recovery. For a second order system of the form (1.1), where $M, K > 0$ and $D \geq 0$, compute a reduced system of the form (1.2), where $\widetilde{M}, \widetilde{K} > 0$ and $\widetilde{D} = \widetilde{D}^\top$.

Inputs: Matrices $M, K > 0$, $D \geq 0$ and B, a tolerance value **tol** > 0.

Outputs: Matrices $\widetilde{M}, \widetilde{K} > 0$, $\widetilde{D} = \widetilde{D}^\top$, and $\widetilde{B}$, gap metric error bound **err**.

1) Start with the reduction by postitive real balanced truncation as presented in Section 3:
   a) Use the sparse Cholesky factors G and H of M, K to derive the first order representation (1.3).
   b) Compute the low rank Cholesky factor $L^\top L = P_{\min}$. One can use for example the Lur'e solver, introduced in [28], that computes low rank factors of a $P > 0$ from a *stabilizing solution triple* $(P, \mathcal{K}, \mathcal{L}) \in \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{m\times 2n} \times \mathbb{R}^{m\times m}$ of the so called *Lur'e equation*

$$\begin{bmatrix} \mathcal{A}^\top P + P\mathcal{A} & P\mathcal{B} - \mathcal{C}^\top \\ \mathcal{B}^\top P - \mathcal{C} & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{K}^\top \\ \mathcal{L}^\top \end{bmatrix} \begin{bmatrix} \mathcal{K} & \mathcal{L} \end{bmatrix} = 0, \tag{8.1}$$

   where the term stabilizing refers to the rank condition

$$\operatorname{rank} \begin{bmatrix} \lambda I_{2n} + \mathcal{A} & \mathcal{B} \\ \mathcal{K} & \mathcal{L} \end{bmatrix} = n + \operatorname{rank} \begin{bmatrix} \mathcal{K} & \mathcal{L} \end{bmatrix} \quad \text{for all} \;\; \lambda \in \mathbb{C}^+.$$

   c) Take the partitioned eigendecomposition

$$L\mathscr{S}_n L^\top = \begin{bmatrix} U_1^- & U_2 & U_1^+ \end{bmatrix} \begin{bmatrix} -\Sigma_1^- & 0 & 0 \\ 0 & \mathscr{S}_{n-r}\Sigma_2 & 0 \\ 0 & 0 & \Sigma_1^+ \end{bmatrix} \begin{bmatrix} (U_1^-)^\top \\ (U_2)^\top \\ (U_1^+)^\top \end{bmatrix}, \tag{8.2}$$

   and define reduction matrices $W^\top := \Sigma_1^{-\frac{1}{2}} \mathscr{S}_r U_1^\top L$ and $V := \mathscr{S}_n L^\top U_1 \Sigma_1^{-\frac{1}{2}}$, where $\Sigma_1 := \operatorname{diag}(\Sigma_1^-, \Sigma_1^+)$ and $U_1 := \begin{bmatrix} U_1^- & U_1^+ \end{bmatrix}$, to compute a reduced first order model

$$\dot{\widetilde{x}}(t) = \widetilde{\mathcal{A}}\widetilde{x}(t) + \widetilde{\mathcal{B}}u(t), \quad \widetilde{y}(t) = \widetilde{\mathcal{C}}\widetilde{x}(t), \tag{8.3}$$

   where $\widetilde{\mathcal{A}} := W^\top \mathcal{A} V$, $\widetilde{\mathcal{B}} := W^\top \mathcal{B}$ and $\widetilde{\mathcal{C}} := \mathcal{C}V$. The gap metric error bound **err** is twice the sum of the diagonal entries of Σ_2.
2) Continue with the second order structure recovery from Section 5 and Section 6.

a) If necessary, then apply a block orthogonal state space transformation as in Remark 5.5 to derive a realization $[\widetilde{\mathcal{A}}_\mathrm{b}, \widetilde{\mathcal{B}}_\mathrm{b}, \widetilde{\mathcal{C}}_\mathrm{b}]$ of the form (5.7) with, additionally, $p_1 = p_2 = \widetilde{k}$.

b) For $\widetilde{\mathcal{A}}_\mathrm{z} := \left[\begin{smallmatrix} \widetilde{\mathcal{A}}_{22} & \widetilde{\mathcal{A}}_{23} \\ -\widetilde{\mathcal{A}}_{23}^\top & \widetilde{\mathcal{A}}_{33} \end{smallmatrix}\right] \in \mathbb{R}^{2\widetilde{k}\times 2\widetilde{k}}$, compute $\widetilde{T} \in \mathrm{Gl}_{2\widetilde{k}}(\mathbb{R})$ such that

$$\widetilde{T}^{-1}\widetilde{\mathcal{A}}_\mathrm{z}\widetilde{T} = \mathrm{diag}(\breve{\mathcal{A}}_{22}, \breve{\mathcal{A}}_{33})$$

with matrices $\breve{\mathcal{A}}_{22} = \mathrm{diag}(\widetilde{\mu}_1, \ldots, \widetilde{\mu}_{2\widetilde{k}})$ and $\breve{\mathcal{A}}_{33} = \mathrm{diag}(\mathcal{P}_{\sigma_1,\tau_1}, \ldots, \mathcal{P}_{\sigma_c,\tau_c})$, where $\mathcal{P}_{\sigma_1,\tau_1} = \left[\begin{smallmatrix} \sigma_i & \tau_i \\ -\tau_i & \sigma_i \end{smallmatrix}\right]$. Perform a state space transformation with the matrix $\mathrm{diag}(I_{m+\ell}, \widetilde{T}, I_{m+\ell})$ to derive a realization $[\breve{\mathcal{A}}, \breve{\mathcal{B}}, \breve{\mathcal{C}}]$ that we partition as

$$\breve{\mathcal{A}} = \begin{bmatrix} 0 & 0 & 0 & \breve{\mathcal{A}}_{14} \\ 0 & \breve{\mathcal{A}}_{22} & 0 & \breve{\mathcal{A}}_{24} \\ 0 & 0 & \breve{\mathcal{A}}_{33} & \breve{\mathcal{A}}_{34} \\ -\breve{\mathcal{A}}_{14}^\top & \breve{\mathcal{A}}_{42} & \breve{\mathcal{A}}_{43} & \breve{\mathcal{A}}_{44} \end{bmatrix}, \; \breve{\mathcal{A}}_{24} = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{2\widetilde{k}} \end{bmatrix}, \; \breve{\mathcal{A}}_{42}^\top = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{2\widetilde{k}} \end{bmatrix}, \tag{8.4}$$

$$\widetilde{\mathcal{A}}_{34}^\top = \begin{bmatrix} \mathbf{c}_1 & \mathbf{d}_1 & \cdots & \mathbf{c}_c & \mathbf{d}_c \end{bmatrix}, \; \breve{\mathcal{A}}_{43} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{f}_1 & \cdots & \mathbf{e}_c & \mathbf{f}_c \end{bmatrix}.$$

c) To retrieve the symmetry structure construct a transformation $\breve{T}$:
   - For $i \in \{1, \ldots, q\}$ choose $j \in \{1, \ldots, m\}$ such that $|a_{i,j}| \cdot |b_{i,j}| > \mathbf{tol}$, where $a_{i,j}, b_{i,j}$ are the jth entry of $\mathbf{a}_i$ and $\mathbf{b}_i$, respectively, and set $\breve{t}_i := \left|\frac{a_{i,j}}{b_{i,j}}\right|$.
   - For $i \in \{1, \ldots, c\}$ we either find some $j \in \{1, \ldots, m\}$ such that for the j'th entries of $\mathbf{c}_i, \mathbf{d}_i, \mathbf{e}_i$, and $\mathbf{f}_i$, which we call $c_{i,j}, d_{i,j}, e_{i,j}, d_{i,j}$, respectively, $|c_{i,j}|, |f_{i,j}|, |e_{i,j}|, |d_{i,j}| > \mathbf{tol}$ or $\sigma_i + \mathrm{i}\tau_i$ is close to being an unobservable or uncontrollable mode in which case $|c_{i,j}|, |f_{i,j}|, |e_{i,j}|, |d_{i,j}| \leq \mathbf{tol}$ for all $j \in \{1, \ldots, m\}$. In the first case set $z_i^2 := \frac{d_i^2 + c_i^2}{e_i^2 + f_i^2}$ and find $\left[\begin{smallmatrix} \widetilde{x}_i \\ \widetilde{y}_i \end{smallmatrix}\right] \neq 0$ which solves
$$\begin{bmatrix} e_i z_i^2 - d_i & -f_i z_i^2 - c_i \\ f_i z_i^2 - c_i & e_i z_i^2 + d_i \end{bmatrix} \begin{bmatrix} \widetilde{x}_i \\ \widetilde{y}_i \end{bmatrix} = 0.$$
     Scale the vector above via $\left[\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right] := \sqrt{\frac{z_i}{\widetilde{x}_i^2 + \widetilde{y}_i^2}} \left[\begin{smallmatrix} \widetilde{x}_i \\ \widetilde{y}_i \end{smallmatrix}\right]$ and set $\breve{T}_i := \left[\begin{smallmatrix} x_i & y_i \\ -y_i & x_i \end{smallmatrix}\right]$.
   - If for all $j \in \{1, \ldots, m\}$ $|a_{i,j}| \cdot |b_{i,j}| \leq \mathbf{tol}$ or $|c_{i,j}|, |f_{i,j}|, |e_{i,j}|, |d_{i,j}| \leq \mathbf{tol}$ we can freely choose $\breve{t}_1 = 1$ and $\breve{T}_i = I_2$.

   Define the transformation $\breve{T} := \mathrm{diag}(I_{m+\ell}, \breve{t}_1, \ldots, \breve{t}_{2\widetilde{k}}, \breve{T}_1, \ldots, \breve{T}_c, I_{m+\ell})$ to get a realization $[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}, \widehat{\mathcal{C}}] := [\breve{T}^{-1}\breve{\mathcal{A}}\breve{T}, \breve{T}^{-1}\breve{\mathcal{B}}, \breve{\mathcal{C}}\breve{T}]$. Apply a state space transformation $\breve{V} := \mathrm{diag}(I_{m+\ell+2\widetilde{k}}, \Theta_1, \ldots, \Theta_c, I_{m+\ell})$, where Θ_i is taken from (4.3) and exchange the rows and columns to get a realization $[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}]$ of the form of (5.4).

   Note: The fact that $z_i > 0$ and thus, $\sqrt{z_i} \in \mathbb{R}$ is guaranteed by the following arguments: By following the steps in the proof of *ii)* $\Rightarrow$ *iii)* in Theorem 6.5, we see that there exists a state space transformation for the reduced system that leads to a system of the form $[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}]$. Moreover, such a state space transformation, except from the rows and columns corresponding to unobservable or uncontrollable modes, has to be of the form $\breve{T}\breve{V}$.

d) If the submatrices $\mathcal{A}_{44} = \mathrm{diag}(\widetilde{\mu}_1^+, \ldots, \widetilde{\mu}_{\widetilde{k}}^+)$ and $\mathcal{A}_{55} = \mathrm{diag}(\widetilde{\mu}_1^-, \ldots, \widetilde{\mu}_{\widetilde{k}}^-)$ from $\mathcal{A}_\mathrm{n}$ as in (5.4) do not fulfill $\mathcal{A}_{44} - \mathcal{A}_{55} > 0$: find

$$0 > \widetilde{\mu}_q^+ \geq \ldots \geq \widetilde{\mu}_{\widetilde{k}+1}^+ > \max\{\widetilde{\mu}_{\widetilde{k}}^-, \widetilde{\mu}_{\widetilde{k}}^+\}, \text{ and}$$
$$\widetilde{\mu}_q^- \leq \ldots \leq \widetilde{\mu}_{\widetilde{k}+1}^- < \min\{\widetilde{\mu}_1^-, \widetilde{\mu}_1^+\}$$

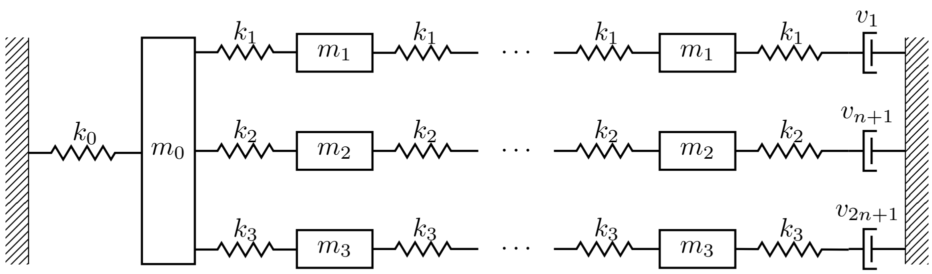

FIG. 8.1. *Triple chain oscillator with* $(3n+1)$ *masses and three dampers.*

and replace $\mathcal{A}_{44}$, $\mathcal{A}_{55}$, $\mathcal{A}_{48}^\top$, $\mathcal{A}_{58}^\top$ with

$$\breve{\mathcal{A}}_{44} := \operatorname{diag}(\mathcal{A}_{44}, \Lambda^+), \quad \breve{\mathcal{A}}_{55} := \operatorname{diag}(\Lambda^-, \mathcal{A}_{55}),$$
$$\breve{\mathcal{A}}_{48}^\top := \begin{bmatrix}\mathcal{A}_{48}^\top & 0\end{bmatrix}, \qquad \breve{\mathcal{A}}_{58}^\top := \begin{bmatrix}0 & \mathcal{A}_{58}^\top\end{bmatrix},$$

where $\Lambda^- = \operatorname{diag}(\widetilde{\mu}_q^-, \ldots, \widetilde{\mu}_{\widetilde{k}+1}^-)$ and $\Lambda^+ = \operatorname{diag}(\widetilde{\mu}_{\widetilde{k}+1}^+, \ldots, \widetilde{\mu}_q^+)$ such that $\breve{\mathcal{A}}_{44} - \breve{\mathcal{A}}_{55} > 0$. Abusing notations we rename $\breve{\mathcal{A}}_{55} = \operatorname{diag}(\widetilde{\mu}_1^-, \ldots, \widetilde{\mu}_q^-)$.

e) For $i = 1, \ldots, q$ set $a_i := \sqrt{\frac{\widetilde{\mu}_i^-}{\widetilde{\mu}_i^- - \widetilde{\mu}_i^+}}$, $b_i := \sqrt{\frac{\widetilde{\mu}_i^+}{\widetilde{\mu}_i^- - \widetilde{\mu}_i^+}}$ and

$$T_i := \begin{bmatrix} a_i & b_i \\ b_i & a_i \end{bmatrix}, \quad T := \operatorname{diag}(I_{m+\ell}, T_1, \ldots, T_{2\widehat{k}}, I_{\ell+m}).$$

Apply the state space transformation T to the system.

f) Suitable block exchanges brings the system into a realization $[\mathcal{A}_\mathrm{s}, \mathcal{B}_\mathrm{s}, \mathcal{C}_\mathrm{s}]$ of the block form (5.1) from Lemma 5.1. A second order realization of $\widetilde{\mathbf{G}}(s)$ is given by:

$$\ddot{\widetilde{p}}(t) + \underbrace{\begin{bmatrix} D_{11} & D_{12} \\ D_{12}^\top & D_{22} \end{bmatrix}}_{=:\widetilde{D}} \dot{\widetilde{p}}(t) + \underbrace{\begin{bmatrix} 0 & G_{21}^\top \\ G_{12}^\top & G_{22}^\top \end{bmatrix} \begin{bmatrix} 0 & G_{12} \\ G_{21} & G_{22} \end{bmatrix}}_{=:\widetilde{K}} \widetilde{p}(t) = \underbrace{\begin{bmatrix} 0 \\ \widehat{\mathcal{B}}_6 \end{bmatrix}}_{=:\widetilde{B}} u(t),$$
$$\widetilde{y}(t) = \widetilde{B}^\top \dot{\widetilde{p}}(t).$$

Remark 8.1. *In the case that the submatrix* $\widetilde{\mathcal{A}}_\mathrm{z}$ *of* $\widetilde{\mathcal{A}}$ *as in* (5.8) *does not have semi-simple eigenvalues, one can perturb the blocks of* $\widetilde{\mathcal{A}}_\mathrm{z}$ *corresponding to positive real characteristic values lower than one. In doing so, if additionally* $\widetilde{\mathcal{C}}\widetilde{\mathcal{A}}\widetilde{\mathcal{B}} > 0$, *the newly formed system* $[\widetilde{\mathcal{A}} + \Delta, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$, *for some sufficiently small* $\|\Delta\|_2$, *will then still be passive, see [4]. An* H^∞*-error bound can be computed.*

We illustrate the performance of the above procedure with an example of three coupled mass-spring-damper chains, see [35, Ex. 2].

Example 8.2. *The triple chain consists of three rows that are coupled via a mass* m_0 *which is connected to the fixed base with a spring with stiffness* k_0. *Each row contains* n *masses,* $n+1$ *springs and one damper. The latter is attached to a wall, see Figure 8.1. One can write the free system as*

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = 0,$$

where $M, D,$ and K are defined as $M = \operatorname{diag}(m_1, \ldots, m_1, m_2, \ldots, m_2, m_3, \ldots, m_3)$ *and* $D = \alpha M + \beta K + v(e_1 e_1^\top + e_{n+1} e_{n+1}^\top + e_{2n+1} e_{2n+1}^\top)$, *where e_i denotes the i'th unit vector in $\mathbb{R}^n$ and with the dampers' viscosity v. Moreover,*

$$K = \begin{bmatrix} K_{11} & & & -\kappa_1 \\ & K_{22} & & -\kappa_2 \\ & & K_{33} & -\kappa_3 \\ -\kappa_1^\top & -\kappa_2^\top & -\kappa_3^\top & k_1 + k_2 + k_3 + k_0 \end{bmatrix}, \quad K_{ii} = k_i \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix},$$

where $\kappa_i = \begin{bmatrix} 0 & \ldots & 0 & k_i \end{bmatrix}^\top \in \mathbb{R}^{1 \times n}$ *and* $K_{ii} \in \mathbb{R}^{n \times n}$ *for* $i = 1, 2, 3$. *We choose the input* $b = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}$ *and equally measure the velocities* $c_{\mathrm{v}} = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^\top$. *The second order control system then reads*

$$M\ddot{p}(t) + D\dot{p}(t) + Kp(t) = bu(t), \quad y(t) = c_{\mathrm{v}}\dot{p}(t).$$

We consider the triple chain with $n = 500$, *thus the number of positions is* $3n + 1 = 1501$, $k_0 = 50$, $k_1 = 10$, $k_2 = 20$, $k_3 = 1$, $m_0 = 1$, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, $\alpha = \beta = 0.002$ *and* $v = 5$. *Following the previously presented numerical procedure we first compute a first order reduced model of order* $2r = 300$. *Applying Step 2) in this procedure, we obtain a second order model of order* $r = 150$. *The latter has the form*

$$\widetilde{M}\ddot{\widetilde{p}}(t) + \widetilde{D}\dot{\widetilde{p}}(t) + \widetilde{K}\widetilde{p}(t) = \widetilde{B}u(t), \quad \widetilde{y}(t) = \widetilde{B}^\top\dot{\widetilde{p}}(t),$$

with symmetric $\widetilde{M}$, $\widetilde{D}$, $\widetilde{K} \in \mathbb{R}^{r \times r}$, *where* $\widetilde{M} = I_r$, $K > 0$ *and D has one negative eigenvalue* $\lambda \approx -3.535 \cdot 10^{-2}$ *and its largest eigenvalue is* $\lambda_{\max} \approx 3.162 \cdot 10^0$. *The sigma plots of the original and reduced transfer function* $\mathbf{G}(s)$ *and* $\widetilde{\mathbf{G}}(s)$ *together are depicted in Figure 8.2(a), whereas Figures 8.2(b) and 8.2(c) show the absolute and relative error of the transfer function evaluated on the imaginary axis. With a maximum relative error of approximately* $4.3 \cdot 10^{-2}$ *we obtain a good match between the original and the reduced second order system.*

9. Conclusion. We have presented a numerical procedure for model reduction of passive second order systems that preserves asymptotic stability, passivity and restores the second order structure. The underlying reduction method is positive real balanced truncation, which provides the gap metric error bound from [17] and is shown to yield a first-order system with a special structure. It is further shown that, under some further conditions, this reduced-order system can be represented as a second-order system.

Acknowledgement. Ines Dorschky thanks Serkan Gugercin for the hospitality and useful advises during the research stay at Virginia Tech.

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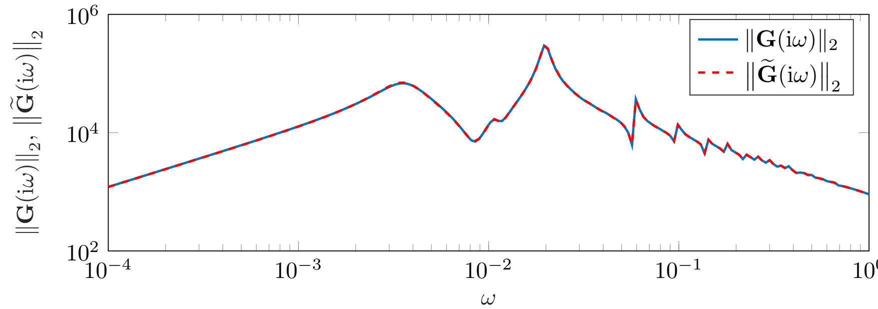
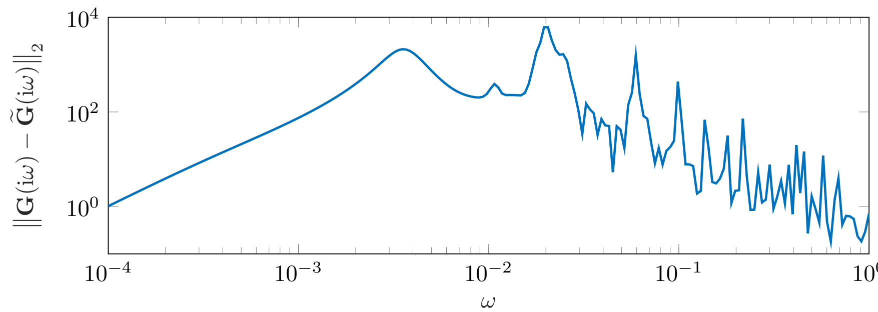
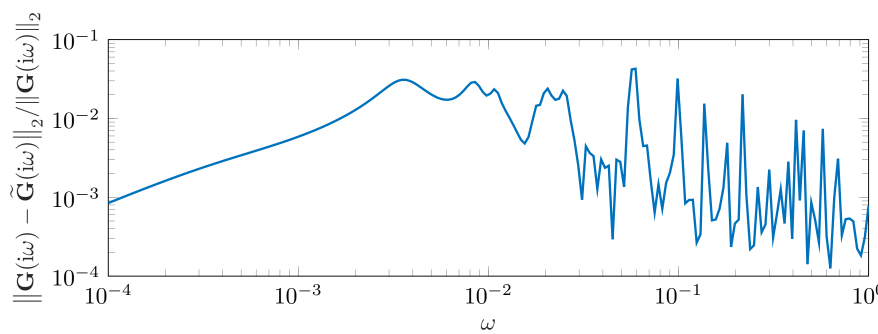


(a) Sigma plots of the original and the reduced transfer functions.



(b) Absolute error.



(c) Relative error.

FIG. 8.2. *Frequency plots of original and reduced transfer functions and errors.*

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