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Benders Decomposition for the Periodic Event Scheduling Problem

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Abstract. The Periodic Event Scheduling Problem (PESP) is the central mathematical model behind the optimization of periodic timetables in public transport. We apply Benders decomposition to the incidence-based MIP formulation of PESP. The resulting formulation exhibits particularly nice features: The subproblem is a minimum cost network flow problem, and feasibility cuts are equivalent to the well-known cycle inequalities by Odijk. We integrate the Benders approach into a branch-and-cut framework, and assess the performance of this method on instances derived from the benchmarking library PESPlib.

Keywords: Periodic Timetabling, Periodic Event Scheduling Problem, Benders Decomposition, Mixed Integer Programming

1 Introduction

Public transport is an important pillar of everyday mobility, and its expansion is indispensable in order to increase the share of climate-friendly traffic. A large part of public transportation networks is operated in a periodic manner, and this creates the need for periodic timetable optimization by mathematical methods. The standard model for this purpose is the Periodic Event Scheduling Problem (PESP), which is difficult to solve, both in theory and practice. We investigate a Benders decomposition approach to PESP, providing a new mixed integer programming formulation. We formally define our setting in Section 2. The Benders reformulation is presented and analyzed in Section 3. We evaluate the method computationally in Section 4.

2 The Periodic Event Scheduling Problem

2.1 Problem Definition

The input to the Periodic Event Scheduling Problem is given by a 5-tuple (G, T, ℓ, u, w), where

- $G = (V, E)$ is a directed graph,
- $T \in \mathbb{N}$ is a period time,

- $\ell \in \mathbb{R}^E_{\geq 0}$ is a vector of lower bounds,
- $u \in \mathbb{R}^E_{\geq 0}$ is a vector of upper bounds, $u \geq \ell$,
- $w \in \mathbb{R}^E_{\geq 0}$ is a vector of weights.

A *periodic timetable* is a vector $\pi \in [0, T)^V$ such that there exists a *periodic tension* $x \in \mathbb{R}^E$ such that

$$\forall (i,j) \in E: \ \ell_{ij} \leq x_{ij} \leq u_{ij} \quad \text{and} \quad \pi_j - \pi_i \equiv x_{ij} \mod T. \tag{1}$$

A periodic timetable π assigns times in $[0, T)$ to the vertices in V, a periodic tension x fixes arc durations within the bounds, and constraint (1) ensures the compatibility of π and x modulo the period time T.

Definition 1 ([12]). *Given* (G, T, ℓ, u, w) *as above, the* Periodic Event Scheduling Problem (PESP) *is to find a periodic timetable* π *along with a periodic tension* x *such that* $w^\top x$ *is minimum.*

Equivalently, one may minimize the weighted periodic slack $w^\top (x - \ell)$. If π is a periodic timetable, then a periodic tension x with minimum $w^\top x$ compatible to π can be computed by

$$x_{ij} := [\pi_j - \pi_i - \ell_{ij}]_T + \ell_{ij}, \qquad \text{for all } (i,j) \in E,$$

where $[\cdot]_T$ denotes the modulo T operator taking values in $[0, T)$.

In the context of periodic timetabling in public transport, vertices often model departure or arrival events of vehicles at stations. Arcs represent, e.g., driving or dwelling of vehicles, transfers for passengers, and safety conditions [6]. The weights typically reflect the number of passengers making use of a vehicle or a transfer, so that the PESP objective amounts to minimizing the total travel time of all passengers.

2.2 Incidence-Based MIP Formulation

Let $A \in \{-1, 0, 1\}^{V \times E}$ denote the incidence matrix of G, i.e., the matrix whose columns are the unit vector differences $e_j - e_i$ for $(i,j) \in E$. By (1), a vector x is a periodic tension for a periodic timetable π if and only if $\ell \leq x \leq u$ and $A^\top \pi \equiv x \bmod T$. In particular, we can write $x = A^\top \pi + Tp$ for some integer vector $p \in \mathbb{Z}^E$. This allows to express PESP as the following mixed-integer linear program (MIP), cf. [5, 7, 10]:

$$\begin{aligned} &\text{Minimize} && (Aw)^\top \pi + T w^\top p \\ &\text{s.t.} && \ell \leq A^\top \pi + Tp \leq u \\ &&& \pi \in \mathbb{R}^V \\ &&& p \in \mathbb{Z}^E. \end{aligned} \tag{2}$$

The domain of π can be extended beyond $[0, T)$: for each feasible solution (π, p) to (2), the vector $[\pi]_T$ is a periodic timetable in $[0, T)^V$, $\pi - [\pi]_T \equiv 0 \bmod T$, and $([\pi]_T, p + A^\top(\pi - [\pi]_T)/T)$ has the same objective value as (π, p).

To make (2) even more compact, consider the digraph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ contains all arcs in E and additionally a reverse copy $\overline{e}$ for each arc $e \in E$. Define $c(p)_e := u_e - Tp_e$ and $c(p)_{\overline{e}} := -\ell_e + Tp_e$ for all $e \in E$. Then

$$\ell \leq A^\top \pi + Tp \leq u \quad \Leftrightarrow \quad \overline{A}^\top \pi \leq c(p), \tag{3}$$

where $\overline{A}$ denotes the incidence matrix of $\overline{G}$.

3 Benders Decomposition

We apply a classical Benders decomposition [1] to the MIP formulation (2), considering as Benders subproblem the dual of the linear program (LP) that arises for a fixed vector $p \in \mathbb{Z}^E$.

3.1 Analysis of the Subproblem

Using (3), the Benders subproblem reads

$$\begin{aligned} \text{Maximize} \quad & -c(p)^\top f \\ \text{s.t.} \quad & \overline{A} f = -Aw \\ & f \geq 0. \end{aligned} \tag{4}$$

In this form, (4) is equivalent to an uncapacitated minimum cost flow problem in the network $\overline{G}$ with balance $-Aw$ and cost $c(p)$. This has also been observed in [8] in a different context. In particular, minimum cost flow algorithms can be applied to solve the Benders subproblem rather than general-purpose LP solvers.

Lemma 1. *The Benders subproblem* (4) *is always feasible.*

Proof. By Gale's theorem [3], (4) is feasible if and only if for every subset $S \subseteq V$ the sum of balances $\sum_{v \in S} (-Aw)_v$ is at most the capacity of all arcs leaving S. As capacities are infinite, we only need to consider such S that do not admit any leaving arc. However, as $\overline{G}$ contains for each arc a reverse copy, S can only be a union of connected components of $\overline{G}$ and hence of G. But then the rows of A corresponding to the vertices in S add to 0, so that $\sum_{v \in S} (-Aw)_v = 0$.

We now turn to boundedness of (4). An oriented cycle in G is a vector $\gamma \in \{-1, 0, 1\}^E$ such that $\{e \in E \mid \gamma_e \neq 0\}$ becomes a cycle when undirecting G. Any oriented cycle can be decomposed as $\gamma = \gamma_+ - \gamma_-$, where $\gamma_+ := \max(\gamma, 0)$ is the forward part, and $\gamma_- := \max(-\gamma, 0)$ is the backward part of γ.

Lemma 2. *For $p \in \mathbb{Z}^E$, the following are equivalent:*

a) *The Benders subproblem* (4) *is bounded for p.*
b) *There is no directed cycle in $\overline{G}$ of negative cost w.r.t. $c(p)$.*

c) *For all oriented cycles γ in G holds*

$$\gamma^\top p \leq \left\lfloor \frac{\gamma_+^\top u - \gamma_-^\top \ell}{T} \right\rfloor.$$

Proof. The equivalence of a) and b) is well-known for network flow problems. By LP duality and Lemma 1, a) is equivalent to the feasibility of the LP arising from (2) when fixing p. That the latter is in turn equivalent to b) resp. c) is indicated in [9] and is explained in detail in the proof of Theorem 4.3 in [11].

Remark 1. In the PESP literature, the inequalities in Lemma 2c are known as Odijk's cycle inequalities [9], which are the base for a cutting plane algorithm to construct a feasible, but not necessarily optimal, periodic timetable [10].

3.2 Master Problem

Having discussed the subproblem, we now turn to the master problem:

Theorem 1. *The following mixed integer program solves PESP:*

$$\begin{array}{llr} \text{Minimize} & z & \\ \text{s.t.} & z - Tw^\top p \geq -c(p)^\top f, & f \text{ feasible for (4)}, \\ & c(p)(C) \geq 0, & C \text{ directed cycle in } \overline{G}, \\ & z \in \mathbb{R}, & \\ & p \in \mathbb{Z}^E. & \end{array} \tag{5}$$

An optimal periodic timetable π^* can be recovered from an optimal solution (z^*, p^*) by solving the LP that arises from (2) by fixing p to p^*.

The proof of Theorem 1 is straightforward using the standard Benders decomposition technique [1]. The second line of constraints in (5) corresponds to the Benders feasibility cuts, which, by Lemma 2, are equivalent to Odijk's cycle inequalities for each oriented cycle. The first line of constraints correspond to the Benders optimality cuts. It is sufficient to consider these cuts only for vertices of the polyhedron $\{f \geq 0 \mid \overline{A}f = -Aw\}$, i.e., extremal flows given by spanning tree structures, i.e., flows $f \geq 0$ that can be positive only on the arcs of a spanning tree of $\overline{G}$. This way, the MIP (5) is endowed with a finite description, however, there will in general be exponentially many spanning trees, and exponentially many cycles. When solving the Benders master problem with a MIP solver in practice, it is therefore necessary to generate the constraints dynamically.

Remark 2. Any PESP instance can be preprocessed so that it is no restriction to assume $p \in \{0, 1, 2\}^E$ [5]. This is useful for breaking symmetries in (5).

Remark 3. Spanning trees in $\overline{G}$ provide lower bounds on the optimal objective value by means of the Benders optimality cuts in (5). On the other hand, spanning trees in $\overline{G}$ correspond to spanning trees in G with an additional marking of the tree arcs as either original or reversed. The latter is the combinatorial structure behind the vertices of the periodic tension polytope [7]. In particular, the value $w^\top x$ of any such vertex x is an upper bound on the optimal value.

4 Computational Results

We implemented a branch-and-cut algorithm to solve formulation (5) using the generic callback framework of CPLEX 12.10. The subproblem is solved using the network simplex implementation available in CPLEX. To stabilize and accelerate the solution process, we use the method for cut loop stabilization described in [2]. The computations were carried out on a Dell Precision 7520 running Windows 10 with an Intel Core i7-7820HQ processor at 2.9 GHz and with 16 GB of RAM.

We test the Benders approach on sub-instances of `R1L1`, one of the instances from the benchmark library PESPlib [4]. Table 1 presents the objective (weighted slack), optimality gap and number of cuts, obtained with a computation time of 20 minutes. We find that the Benders approach terminates with a large optimality gap for all instances. The resulting solution for instance `R1L1-0.8` is known to be optimal, but it appears that the Benders optimality cuts are not strong enough to close the optimality gap. For the other instances, the found solutions are worse than best known solutions, hence the Benders approach is not competitive with other approaches, neither on the primal nor on the dual side. The number of generated cuts for all instances is very large, indicating that the cuts are relatively weak.

Table 1. Results of the branch-and-cut algorithm.

Instance	Objective	Optimality Gap (%)	Cuts
`R1L1-0.8`	1 032 021	44.1	3 743
`R1L1-0.7`	3 568 074	81.4	30 828
`R1L1-0.6`	9 080 015	82.8	23 925

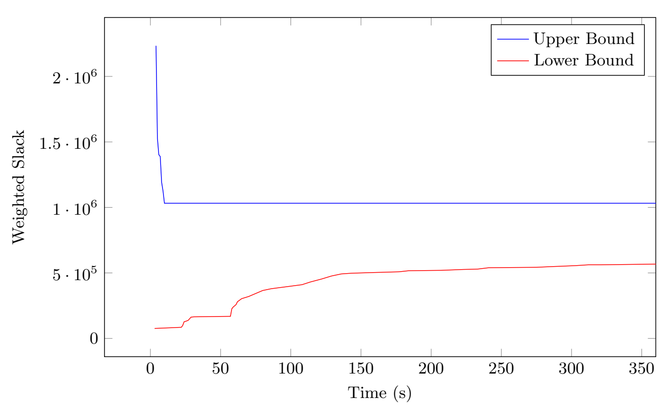

Fig. 1. Evolution of the lower and upper bound for instance `R1L1-0.8`.

The typical behavior of the Benders approach is illustrated in Figure 1, visualizing the evolution of the lower and upper bounds on the instance `R1L1-0.8`. We observe that the optimal solution is already found within 10 seconds. On the other hand, there is a large gap between lower and upper bound, with the lower bound increasing at a diminishing rate. This is rather disappointing, as the underlying digraph G of `R1L1-0.8` has only 23 nodes and 36 arcs, and the incidence-based MIP formulation (2) can be solved to optimality by CPLEX within less than a second.

We therefore conclude that Benders decomposition, despite its attractive theoretical properties, does not seem to be beneficial compared to the well-established solution methods for the PESP.

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